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Classification of stable homotopy types with torsion-free homology

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Abstract

We compute the number of indecomposable stable homotopy types with finitely generated torsion free homology of stable dimension $k \geqslant 0$. © 2001 Elsevier Science Ltd. All rights reserved.

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A polyhedron is a triangulate compact subspace of an Euclidean space $\mathbb{R}^N$, $N \geqslant 0$. The principal idea of homotopy type is to consider spaces which are deformed or perturbed continuously a little bit as being similar. The equivalence relation generated by such slight perturbation has its precise definition by the notion of homotopy equivalence. A class of homotopy equivalent polyhedra X is termed a *homotopy type* $\{X\}$.

The classification of homotopy types is the classical and fundamental task of algebraic topology. J.H.C. Whitehead's seminal work focused mainly on this problem. Moreover, surgery theory showed that it is necessary to classify homotopy types in order to classify diffeomorphism types of closed manifolds. In fact, homotopy types are archetypes underlying most geometric structures.

The main numerical invariants of a homotopy type are dimension and degree of connectedness. The complexity of a homotopy type in general is largely measured by the dimension and, for

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a given dimension, the complexity diminishes as the connectivity increases. A classical result of J.H.C. Whitehead is the following proposition which classifies a certain class of homotopy types completely.

Proposition. *An $(n-1)$-connected n-dimensional polyhedron X has the homotopy type of a one-point union of n-dimensional spheres*, $n \geqslant 1$.

The proof is an application of theorems learned by each student of algebraic topology. We recall the argument for $n > 1$ as follows. Since X is n-dimensional the homology $H_n(X)$ is free abelian. On the other hand, since X is $(n-1)$-connected the Hurewicz theorem shows that $H_n(X)$ is isomorphic to the homotopy group $\pi_n(X)$ which therefore is also free abelian. A basis in $\pi_n(X)$ yields a map from a one-point union of n-dimensional spheres to X which induces an isomorphism in homology. Hence by the Whitehead theorem this map is a homotopy equivalence.

In this paper we consider more generally the classification of $(n-1)$-connected $(n+k)$-dimensional homotopy types with torsion free homology in the stable range $k < n$. In this range the Freudenthal suspension theorem shows that the classification depends only on k and not on n. Let $\mathbf{F}_n^k$ be the homotopy category consisting of all $(n-1)$-connected $(n+k)$-dimensional CW-complexes with finitely generated torsion free homology. In the range $k < n-1$ the category $\mathbf{F}^k = \mathbf{F}_n^k$ is an additive category which does not depend on n.

The coproduct $X \vee Y$ which is also the product in the additive category $\mathbf{F}^k$ is given by the one-point union of spaces X and Y. A polyhedron A or its homotopy type $\{A\}$ in $\mathbf{F}^k$ is termed *elementary* or *indecomposable* if a homotopy equivalence $A \simeq X \vee Y$ implies that X or Y are homotopy equivalent to a point. All homotopy types of $\mathbf{F}^k$ are obtained by finite one-point unions of such elementary polyhedra in $\mathbf{F}^k$.

A best possible solution for the classification of homotopy types in $\mathbf{F}^k$ is a complete list of elementary homotopy types. For example by the proposition above the sphere S^n is the only indecomposable object in $\mathbf{F}^0$. It is also easy to see that the spheres S^n, S^{n+1} are the only indecomposable objects in $\mathbf{F}^1$.

Whitehead [17] and Chang [8] classified all $(n-1)$-connected $(n+2)$-dimensional homotopy types, $n > 2$, by an explicit list of elementary polyhedra which are discussed in detail in the books of Hilton [12,13]. The objects with torsion free homology in this list are

$$S^n, S^{n+1}, S^{n+2}, S^n \cup_\eta e^{n+2} \in \mathbf{F}^2,$$

which are the indecomposable homotopy types in $\mathbf{F}^2$. Here $\eta = \eta_n \in \pi_{n+1} S^n$ is the Hopf element.

Moreover, Baues–Hennes [7] compute all elementary polyhedra which are $(n-1)$-connected and $(n+3)$-dimensional, $n > 3$. This yields the complete list of indecomposable objects in $\mathbf{F}^3$ given by

$$S^n, S^{n+1}, S^{n+2}, S^{n+3}, S^n \cup_\eta e^{n+2}, S^{n+1} \cup_\eta e^{n+3}, S^n \cup_{\eta\eta} e^{n+3} \in \mathbf{F}^3$$

Here $\eta\eta = \eta_n \eta_{n+1}$ is the double Hopf map. Recently Baues–Drozd [4] also obtained a complete list of 67 indecomposable objects in $\mathbf{F}^4$. Hence for $k \leqslant 4$ the list of indecomposable homotopy types in $\mathbf{F}^k$ turned out to be finite and we were faced with the challenging problem:

"Determine the maximal k for which the set of all indecomposable homotopy types in $\mathbf{F}^k$ is a finite set!"

We compute the number of objects in this set for all k as follows:

Theorem A. *The number* $I(\mathbf{F}^k)$ *of indecomposable homotopy types in* $\mathbf{F}^k$ *is given by the table*

k	0	1	2	3	4	5	$k \geqslant 6$
			…				
$I(\mathbf{F}^k)$	1	2	4	7	67	451	∞.

Below we describe the 451 elementary polyhedra of $\mathbf{F}^5$ by use of "lightning flashs". We would like to point out a wonderful test for the correctness of theorem A given by Spanier–Whitehead duality. Since our proof is inductive by dimension the proof is not at all compatible with Spanier–Whitehead duality. Our result, however, which describes explicitly all 451 elementary polyhedra of $\mathbf{F}^5$ has to satisfy Spanier–Whitehead duality and it does.

Recall that the *isomorphism class group* $K_0(\mathbf{F}^k)$ is the abelian group generated by the homotopy types $\{X\}$ in $\mathbf{F}^k$ with the relations

$$\{X\} + \{Y\} = \{X \vee Y\}.$$

It is a remarkable result of Freyd [10] that $K_0(\mathbf{F}^k)$ is actually a countably generated free abelian group for all $k \geqslant 0$. We compute $K_0(\mathbf{F}^k)$ as an abelian group for all $k \geqslant 0$ as follows:

Theorem B.

k	0	1	2	3	4	5	$k \geqslant 6$
			…				
$K_0(\mathbf{F}^k)$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^7$	$\mathbb{Z}^{29}$	$\mathbb{Z}^{87}$	$\mathbb{Z}^{\infty}$.

Here $\mathbb{Z}^{\infty}$ denotes the free abelian group freely generated by the set $\mathbb{N} = \{1,2,\dots\}$ of natural numbers. We describe a list of generators of $K_0(\mathbf{F}^5)$ in (2.10) below. Theorem A, B and [4] imply the following result concerning the representation type of the additive category $\mathbf{F}^k$.

Theorem C. *The category* $\mathbf{F}^k$ *has finite representation type if and only if* $k \leqslant 5$. *Moreover for* $k \geqslant 10$ *the category* $\mathbf{F}^k$ *has wild representation type.*

Hence only the representation types of $\mathbf{F}^6, \mathbf{F}^7, \mathbf{F}^8, \mathbf{F}^9$ are unknown. The computation of the representation type of $\mathbf{F}^6$ involves a matrix problem given by a matrix with 328 rows and columns; see Section 7.

We are grateful to the referee for his careful remarks which improved the presentation of this paper.

1. The list of elementary polyhedra

We need the following elements in stable homotopy groups of spheres, compare Toda [14]. Let $\eta_n = \eta \in \pi_{n+1}(S^n) = \mathbb{Z}/2$ be the Hopf map, and let $\eta^2 \in \pi_{n+2}(S^n) = \mathbb{Z}/2$ be the double Hopf map, $n \geqslant 3$. Moreover let $v = v_n, \alpha = \alpha_n \in \pi_{n+3}(S^n) = \mathbb{Z}/24$ be the generator of order 8 and 3, respectively. We use stable notation and we write $\pi_k^S = \pi_{n+k}(S^n)$ for $k < n-1$ so that π_*^S is the ring of stable homotopy groups of spheres. As usual we write $|\xi| = k$ if $\xi \in \pi_k^S$.

1.1. Definition. A *lightning flash*

$$g = (\alpha_1 \ldots \alpha_k)_n^\varepsilon$$

is a word of elements $\alpha_i \in \pi_*^S$ with $i = 1,2,\ldots,k$ $(k \geqslant 0)$ together with a number $n \in \mathbb{Z}$ and an element $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$.

We visualize $g = (\alpha_1 \ldots \alpha_k)_n^\varepsilon$ by a simplicial arc in $\mathbb{R}^2$ which has the shape of a lightning flash as in the examples of definition (1.7) below. For this we choose $t_i \in \mathbb{R}$ with $t_0 < t_1 < \cdots < t_k$ and $m_i \in \mathbb{Z}$ with $|m_{i-1} - m_i| = |\alpha_i| + 1$ for $i = 1,\ldots,k$ such that $n = \min\{m_i\}$ and

$$\begin{aligned} &m_0 < m_1 > m_2 < m_3 > \cdots \quad \text{for } \varepsilon = 0,\\ &m_0 > m_1 < m_2 > m_3 < \cdots \quad \text{for } \varepsilon = 1. \end{aligned} \tag{1}$$

Hence all numbers m_i are uniquely determined by g. The simplicial arc g connects successively the sequence of points $(t_0, m_0), (t_1, m_1), \ldots, (t_k, m_k)$ in $\mathbb{R}^2$. Then n is the minimum "level" of a vertex in the arc g.

We associate with a lightning flash g a map between one-point unions of stable spheres as follows. For a sequence $m = (m_0, \ldots, m_k)$ let $S^m = S^{m_0} \vee \cdots \vee S^{m_k}$ be the corresponding one-point union of spheres and let $\Sigma^{-1} S^m$ be the desuspension of S^m. Let $(m|\text{even}) = (m_0, m_2, \ldots)$, resp. $(m|\text{odd}) = (m_1, m_3, \ldots)$ be the subsequences of m given by even, resp. odd, indices. Then the map g is a map

$$\begin{aligned} &g : \Sigma^{-1} S^{(m|\text{even})} \to S^{(m|\text{odd})} \quad \text{for } \varepsilon = 1,\\ &g : \Sigma^{-1} S^{(m|\text{odd})} \to S^{(m|\text{even})} \quad \text{for } \varepsilon = 0. \end{aligned} \tag{2}$$

These maps are defined on spheres in the same way as indicated by the simplicial arc; that is the matrix $(\alpha_{r,s})$ associated to g with $r, s \geqslant 1$ satisfies $\alpha_{r,r} = \alpha_{2r+1}$, and $\alpha_{r+1,r} = \alpha_{2r}$ for $\varepsilon = 0$, and $\alpha_{r,r+1} = \alpha_{2r}$ for $\varepsilon = 1$, and $\alpha_{r,s} = 0$ otherwise.

1.2. Definition. A *lightning flash space* $X(g)$ is the mapping cone of the map g associated to a lightning flash g.

The vertices of the arc g correspond to the non trivial cells of $X(g)$. For example we have for $v \in \pi_3^S, \eta^2 \in \pi_2^S, \eta \in \pi_1^S$

$$X(\eta^2 v \eta)_1^1 = S^{n+1} \vee S^{n+3} \cup_{\eta^2} e^{n+4} \cup_{i_1 v + i_2 \eta} e^{n+5}.$$

If $|\alpha_i| > 0$ for all i then $X(g)$ has torsion free homology and the vertices of the arc g yield the basis of $H_*(X(g))$.

1.3. Remark. We observe that the arc $(\alpha_1 \ldots \alpha_k)_n^0$ is obtained by reflection of the arc $(\alpha_1 \ldots \alpha_k)_n^1$ at a horizontal line. Moreover, we obtain by reflection of $g = (\alpha_1 \ldots \alpha_k)_n^\varepsilon$ at a vertical line the arc $g' = (\alpha_k \alpha_{k-1} \ldots \alpha_1)_n^{k+\varepsilon}$ for which $X(g)$ is homotopy equivalent to $X(g')$. Therefore we identify g and g'. For $k = 0$ we also write $()_n^\varepsilon = S^n$ and for $k = 1$ we write $(\xi)_n = (\xi)_n^0 = (\xi)_n^1$ with $\xi \in \pi_*^S$.

1.4. Definition. For $g = (\alpha_1 \ldots \alpha_k)_n^\varepsilon$ the *dual* $D(g)$ is defined by

$$D(g) = (\alpha_k \alpha_{k-1} \ldots \alpha_1)_n^{k+\varepsilon+1}.$$

The arc of $D(g)$ is obtained by turning the arc g around. The *suspension* Σg is defined by

$$\Sigma(g) = (\alpha_1 \ldots \alpha_k)_{n+1}^\varepsilon.$$

Hence the arc Σg is obtained by moving g one level up.

One readily checks the following lemma; see Baues [3] and (2.3) (2) below.

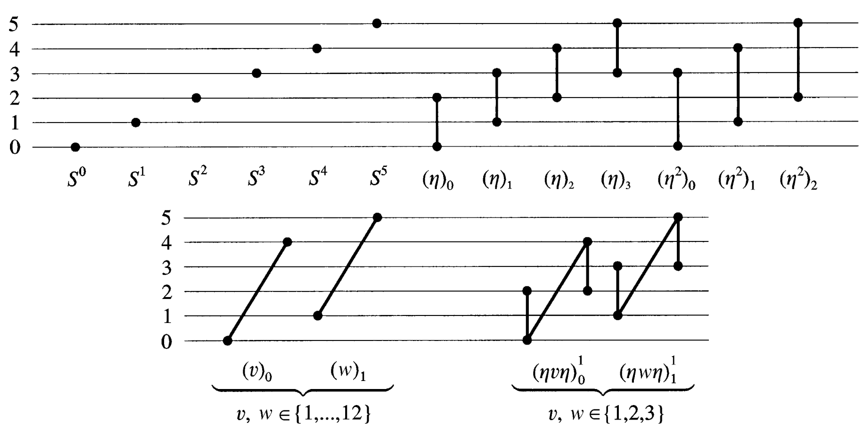
1.5. Lemma. *The suspension $\Sigma X(g)$ of $X(g)$ satisfies $\Sigma X(g) = X(\Sigma g)$. Moreover the Spanier–Whitehead dual $DX(g)$ of $X(g)$ satisfies $DX(g) = X(Dg)$.*

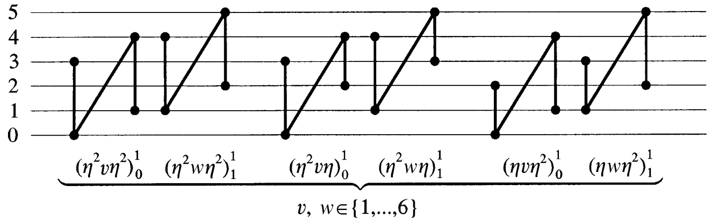
1.6. Remark. The space $X(g)$ is indecomposable if and only if $DX(g)$ is indecomposable. Moreover, if $X(g)$ is indecomposable then all letters α_i in g are nontrivial; the converse, however, does not hold in general. For example, it is known that for $2 \in \mathbb{Z} = \pi_0^S$ and $\eta^2 \in \pi_2^S$ the space $X(2\eta^2 2)_0^1$ is decomposable; see 3.1 (ii) [7].

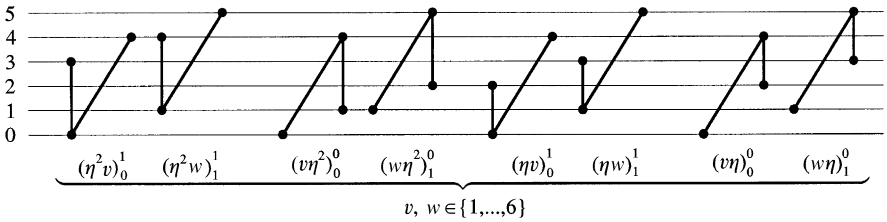
We are now ready to define the list of elementary polyhedra in Theorem A above.

1.7. Definition. The list $\mathscr{L}_5$ of 451 elementary polyhedra is given by lightning flash spaces or words as in the following list where $v, w \in \{1,2,\ldots,24\} = \mathbb{Z}/24 = \pi_3^S$. Here we identify the number v with the element $v(\nu + \alpha)$ where ν and α are the generators of π_3^S described above.

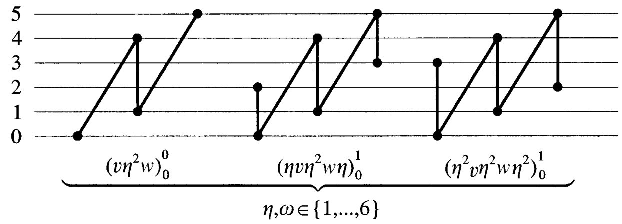
Elementary polyhedra from dimension 4:

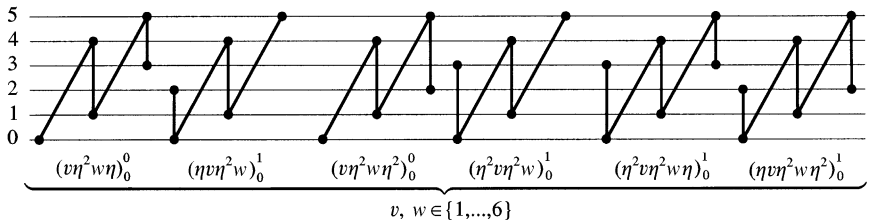






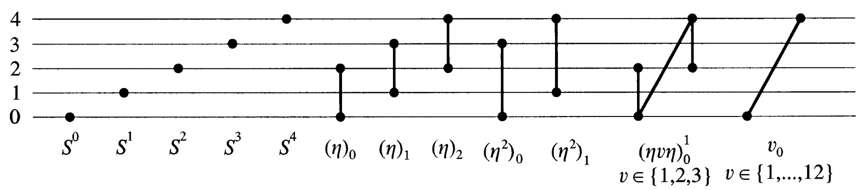
New elementary polyhedra of dimension 5:

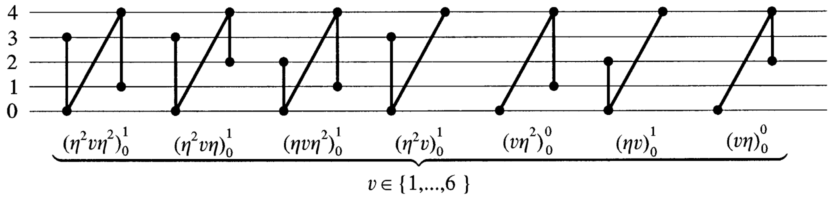




Only the five-dimensional words correspond to graphs which have vertices in level 0 and level 5. Let $\mathscr{L}_4$ be the subset of all graphs in $\mathscr{L}_5$ which have no vertex of level 5. Then $\mathscr{L}_4$ coincides with the following list of 67 elements. (The list $\mathscr{L}_4 = \mathscr{L}$ was already achieved in our previous paper [2] and is needed in the proof below.)

Elementary polyhedra of dimension 4:





Hence the set $\mathscr{L}_5$ is the union of the subsets $\mathscr{L}_4, \Sigma\mathscr{L}_4$ and the set of five-dimensional words.

Though lightning flash spaces are very special spaces we show in Theorem A and in Theorem (2.5) below that each $(n-1)$-connected $(n+5)$-dimensional CW-complex X with finitely generated torsion free homology $(n \geqslant 6)$ is homotopy equivalent to a one-point union of such spaces.

1.8. Remark. Chang and Whitehead [8,17] show that each $(n-1)$-connected $(n+2)$-dimensional CW-complex X with finitely generated homology groups $(n \geqslant 3)$ is homotopy equivalent to a one-point union of the following lightning flash spaces where r is a power of a prime and p, q are powers of 2 with $r, p, q \in \mathbb{Z} = \pi_0^S$ and $\eta \in \pi_1^S$.

$$\begin{cases} S^0,\ S^1,\ S^2,\ (r)_0,\ (r)_1,\ (\eta)_0, \\ (\eta q)_0^0,\ (p\eta)_0^1,\ (p\eta q)_0^1. \end{cases}$$

These are the elementary Chang complexes discussed in the books of Hilton [12,13] and in [3]. In [7] all $(n-1)$-connected $(n+3)$-dimensional CW-complexes with finitely generated homology groups $(n \geqslant 4)$ are classified by certain generalizations of lightning flash spaces. The Chang–Whitehead result [8, 17] was generalised for odd primes p and p–local spaces by Henn [11].

2. Decomposition and congruence of spaces with torsion-free homology

Let $\mathbf{C}$ be an additive category with zero object $*$ and biproducts $A \oplus B$. An object X in $\mathbf{C}$ is *decomposable* if there exists an isomorphism $X \cong A \oplus B$ where A and B are not isomorphic to $*$. A *decomposition* of X is an isomorphism

$$X = A_1 \oplus \cdots \oplus A_n, \quad n < \infty. \tag{2.1}$$

where A_i is indecomposable for all $i \in \{1, \ldots, n\}$. The decomposition of X is *unique* if $B_1 \oplus \cdots \oplus B_m \cong X \cong A_1 \oplus \cdots A_n$ implies that $m = n$ and that there is a permutation σ with $B_{\sigma(i)} \cong A_i$. The *decomposition problem* in $\mathbf{C}$ can be described by the following task: find a complete list of indecomposable isomorphism types in $\mathbf{C}$ and describe the possible decompositions of objects in $\mathbf{C}$. This problem is considered by representation theory. We say that the decomposition problem in $\mathbf{C}$ is wild or equivalently that $\mathbf{C}$ has *wild representation type* if the solution of the decomposition problem would imply a solution of the following problem.

2.2. Problem. Let k be a field and consider the following additive category $\mathbf{V}^{\alpha,\beta}$. Objects are finite-dimensional k-vector spaces V together with two endomorphisms $\alpha_V, \beta_V : V \to V$. Morphisms are k-linear maps $f : V \to W$ satisfying $f\alpha_V = \alpha_W f$ and $f\beta_V = \beta_W f$. The decomposition problem in $\mathbf{V}^{\alpha,\beta}$ for any field k is termed a "wild problem of representation theory".

If the list of all indecomposable objects of **C** is finite then **C** has *finite* representation type. If the representation type of **C** is neither finite nor wild then **C** is of *tame representation type*. In representation theory there are in general means to compute an explicit list of all indecomposable objects in **C** if **C** has finite or tame representation type.

Next, we describe our decomposition problem of homotopy theory. Let $\mathbf{Top}^*/\simeq$ be the homotopy category of pointed topological spaces. The set of morphisms $X \to Y$ in $\mathbf{Top}^*/\simeq$ is the set of homotopy classes $[X, Y]$. Isomorphisms in $\mathbf{Top}^*/\simeq$ are called homotopy equivalences and isomorphism types in $\mathbf{Top}^*/\simeq$ are homotopy types. Let $\mathbf{F}_n^k$ be the full subcategory of $\mathbf{Top}^*/\simeq$ consisting of $(n-1)$-connected $(n+k)$-dimensional CW-complexes which have finitely generated torsion free homology. The objects of $\mathbf{F}_n^k$ are special A_n^k-polyhedra, see [17]. The suspension Σ gives us a sequence of functors

$$\mathbf{F}_1^k \xrightarrow{\Sigma} \mathbf{F}_2^k \to \cdots \to \mathbf{F}_n^k \xrightarrow{\Sigma} \mathbf{F}_{n+1}^k \to \cdots \tag{2.3}$$

with $k \geqslant 0$. The Freudenthal suspension theorem shows that the sequence stabilizes in the sense that for $k+1 < n$ the functor $\Sigma : \mathbf{F}_n^k \to \mathbf{F}_{n+1}^k$ is an equivalence of *additive* categories so that

$$\mathbf{F}^k = \mathbf{F}_n^k \quad \text{with } k+1 < n \tag{3}$$

does not depend on n. See [3]. This is the stable homotopy category of (-1)-connected k-dimensional spectra with finitely generated torsion free homology. There is, however, no need to use the more sophisticated notion of spectrum in this paper. The biproduct in the additive category $\mathbf{F}^k$ is the one-point union of spaces. We point out that for $k+1 = n$ the functor Σ is full and a 1–1 correspondence of homotopy types. See [3]. The *Spanier–Whitehead duality* is a contravariant functor

$$D : \mathbf{F}^k \to \mathbf{F}^k, \tag{4}$$

satisfying $DD = 1$ and $D(S^{n+i}) = S^{n+k-i}$ for $i \in \{0, \dots, k\}$; compare for example [9].

Each nontrivial element α in the $(k-1)$-stem, $\alpha \in \pi_{n+k-1}(S^n)$, yields the canonical 2-*cell complex* $S^n \cup_\alpha e^{n+k} \in \mathbf{F}_n^k$, $n \geqslant 2$, which is indecomposable. Hence, elements in homotopy groups of spheres can essentially be identified with special indecomposable objects in $\mathbf{F}_n^k$, $k \geqslant 2$. The decomposition in $\mathbf{F}_n^k$ is not unique. For example Freyd [10] points out that for $n \geqslant 5$ there is a homotopy equivalence.

$$S^n \vee (S^n \cup_\nu e^{n+4}) \simeq S^n \vee (S^n \cup_{3\nu} e^{n+4}) \tag{2.4}$$

in $\mathbf{F}_n^4$ where, however, the CW-complexes $S^n \cup_\nu e^{n+4}$ and $S^n \cup_{3\nu} e^{n+4}$ are not homotopy equivalent. Here $\nu \in \pi_{n+3}(S^n)$ is a generator of order 8 as in Section 1.

In [4] we solved the decomposition problem in the additive category $\mathbf{F}^4$ by showing that $X(\mathscr{L}_4)$ is a complete list of indecomposable objects in $\mathbf{F}^4$. Our main purpose in this paper is the solution of the decomposition problem in $\mathbf{F}^5$. We show that $\mathbf{F}^5$ is again of finite representation type. On the other hand, we have seen in the appendix of [4] that $\mathbf{F}^k$ is of wild representation type for $k \geqslant 10$. Hence only the representation types of $\mathbf{F}^6$, $\mathbf{F}^7$, $\mathbf{F}^8$, $\mathbf{F}^9$ remain unknown.

2.5. Theorem. *The list* $X(\mathscr{L}_5)$ *of* 451 *elementary polyhedra in Section* 1 *is a complete list of all indecomposable spaces in* $\mathbf{F}^5$. *Hence for* $n \geqslant 6$ *each* $(n-1)$-*connected* $(n+5)$-*dimensional CW-*

complex X with finitely generated torsion free homology admits a homotopy equivalence

$$X \simeq X_1 \vee \cdots \vee X_t$$

with $X_i \in X(\mathscr{L}_5)$ for $1 \leqslant i \leqslant t$.

Following Freyd [10] and Cohen [9] 4.26 we use the following notation.

2.6. Definition. We say that two spaces X, Y in $\mathbf{F}^k$ are *congruent* and we write $X \equiv Y$ if (a) or equivalently (b) is satisfied.

(a) There exists a space Z in $\mathbf{F}^k$ such that $X \vee Z \simeq Y \vee Z$ are homotopy equivalent.
(b) There exists a homotopy equivalence $X \vee B_X \simeq Y \vee B_X$ where B_X is the unique one-point union of spheres which has the same Betti numbers as X, that is $H_*(X)/\text{torsion} = H_*(B_X)$.

2.7. Definition. Let p be a prime. A space X in $\mathbf{F}^k$ is a *p-primary space* if there exists a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[rr, "p^N.1_X"] \arrow[dr] & & X \\ & B \arrow[ur] & \end{tikzcd}$$

where B is a one-point union of spheres. Here p^N is a power of the prime p and $p^N \cdot 1_X$ is a multiple of the identity of X in the abelian group of homotopy classes $[X, X]$ in $\mathbf{F}^k$ and N cannot be chosen to be $N = 0$. This implies that X is not a one-point union of spheres.

2.8. Lemma. *An elementary polyhedron $X(g)$ with $g \in \mathscr{L}_5$ is 2-primary if and only if g is a word with letters in the set $\{\eta, \eta^2, v, w$ with v and w divisible by $3\}$. The only congruences between 2-primary polyhedra in $X(\mathscr{L}_5)$ are given by (2.4) that is $X(3)_0 \equiv X(9)_0$ and $X(3)_1 \equiv X(9)_1$. Moreover, $X(g)$ is 3-primary if and only if $g = (8)_0$ or $g = (8)_1$. For a prime $p > 3$ there are no p-primary spaces in $X(\mathscr{L}_5)$.*

Recall that for any small additive category $\mathbf{C}$ (for example $\mathbf{C} = \mathbf{F}^k$, $k \geqslant 0$) we have the *isomorphism class group $K_0(\mathbf{C})$*. This is the abelian group with one generator [A] for each isomorphism class of objects $A \in \mathbf{C}$ with relations $[A] + [B] = [A \oplus B]$. This is just the *Grothendieck group* of $\mathbf{C}$ as defined by Bass [1]. A typical element of $K_0(\mathbf{C})$ is a formal difference $[A] - [B]$ with $[A] - [B] = [A'] - [B']$ if and only if there exists an isomorphism in $\mathbf{C}$ of the form $A \oplus B' \oplus C \cong A' \oplus B \oplus C$ for some object C in $\mathbf{C}$. The following result is due to Freyd [10]; see also Cohen [9] 4.44.

2.9. Theorem of Freyd. *Let $k \geqslant 0$. Then $K_0(\mathbf{F}^k)$ is a free abelian group generated by the spheres in $\mathbf{F}^k$ and by the congruence classes of indecomposable p-primary spaces in $\mathbf{F}^k$ where p runs through all primes.*

Such a wonderful result yields the crucial task to compute the generators of $K_0(\mathbf{F}^k)$ explicitly. For the category $\mathbf{F}^k$ of torsion-free polyhedra we get accordingly:

- $K_0(\mathbf{F}^0) = \mathbb{Z}$ generated by S^n,
- $K_0(\mathbf{F}^1) = \mathbb{Z}^2$ generated by S^n, S^{n+1},
- $K_0(\mathbf{F}^2) = \mathbb{Z}^4$ generated by S^n, S^{n+1}, S^{n+2}, $X((\eta)_0)$,
- $K_0(\mathbf{F}^3) = \mathbb{Z}^7$ generated by S^n, S^{n+1}, S^{n+2}, S^{n+3}, and $X((\eta)_0)$, $X((\eta)_1)$, $X((\eta^2)_0)$.

In [4] we show that

- $K_0(\mathbf{F}^4) = \mathbb{Z}^{29}$

is generated by the 5-spheres $S^n, \dots, S^{n+4}$ in $\mathbf{F}^4$, the 23 congruence classes of 2-primary polyhedra in $X(\mathscr{L}_4)$ and by the unique 3-primary polyhedron in $X(\mathscr{L}_4)$.

Using (2.8) and (2.5) we get accordingly:

2.10. Theorem. *The group $K_0(\mathbf{F}^5) = \mathbb{Z}^{87}$ is generated by the* 6*-spheres $S^n, \dots, S^{n+5}$, the two* 3*-primary polyhedra in $X(\mathscr{L}_5)$ and the* 79 *indecomposable congruence classes of* 2*-primary polyhedra in $X(\mathscr{L}_5)$.*

In Section 7 we show that $K_0(\mathbf{F}^k) = \mathbb{Z}^\infty$ for $k \geqslant 6$.

3. The algebraic classification of homotopy types in $\mathbf{F}^5$

Let X be a CW-complex in $\mathbf{F}^5 = \mathbf{F}^5_n$ with $n \geqslant 7$. We use *stable notation* so that we are allowed to omit n. Hence S^i corresponds to the sphere S^{n+i}, moreover e^i corresponds to the cell e^{n+i}, and the homotopy group $\pi_j S^i$ corresponds to $\pi_{n+j} S^{n+i}$, etc. We may assume that X has a cell structure given by the homology decomposition; see [3]. This implies (since the homology of X is free abelian) that the cells of X are in 1–1 correspondence with free generators of the homology of X. Let c_i be the number of i-cells in X so that $H_i(X) = \mathbb{Z}^{c_i}$. For a space A and a natural number $d \geqslant 0$ let

$$dA = \underbrace{A \vee \cdots \vee A}_{d\text{-times}} \tag{3.1}$$

be the d-fold one-point union of A. The attaching map of (stable) 5-cells of X is a map

$$f: c_5 S^4 \to X^4 \tag{3.2}$$

with $X^4 \in \mathbf{F}^4$. Here X^4 is the 4-skeleton of X. Since the cell structure is given by a homology decomposition we know that f admits a factorization

$$f: c_5 S^4 \to X^3 \subset X^4 \tag{5}$$

with $X^3 \in \mathbf{F}^3$. Let

$$\Gamma_4 X = \operatorname{image}\{\pi_4 X^3 \to \pi_4 X^4\}.$$

Then the homotopy class of f in (3.2) is determined by a homomorphism

$$f\colon H_5(X) = \mathbb{Z}^{c_5} \to \Gamma_4(X^4). \tag{6}$$

This is the secondary boundary in Whitehead's exact sequence [18]. In [4] we classified the homotopy types in $\mathbf{F}^4$ showing that X^4 is a one-point union of spaces $X(g)$ with $g \in \mathscr{L}_4$. We say that f in (6) is in *normal form* if X is a one-point union of spaces $X(g)$ with $g \in \mathscr{L}_5$ so that f is canonically given by the attaching maps of 5-cells in $X(g)$, see Section 1.

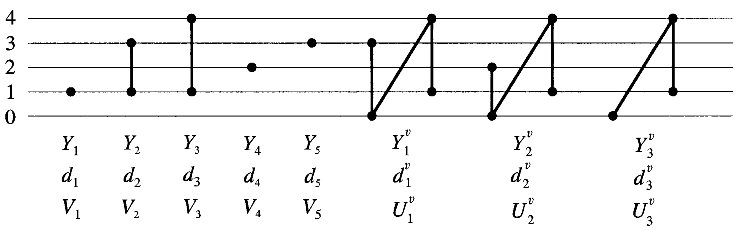
Since we are in the stable range the functor Γ_4 is additive. Therefore, we can compute $\Gamma_4 X^4$ as in proposition (3.4) below which determines $\Gamma_4 X(g)$ for $g \in \mathscr{L}_4$. We define a subset $\mathscr{L}'_4$ of $\mathscr{L}_4$ by

$$\mathscr{L}'_4 = \begin{Bmatrix} S^0, & S^4, & (\eta)_0, & (\eta)_2, & (\eta^2)_0, & (\eta^2 v\eta)^1_0, \\ (\eta v\eta)^1_0, & (\eta^2 v)^1_0, & (\eta v)^1_0, & (v\eta)^0_0, & (v)_0 & \end{Bmatrix}.$$

The proof of our main theorem (2.5) below is highly based on matrix computations. In order to be able to index the entries of such matrices, we choose the following notation compatible with the notation in Baues–Drozd [4]. We define the spaces Y_i and Y^v_j with $i \in \{1,\dots,5\}$, $j \in \{1,2,3\}$, $v \in \{1,\dots,6\}$ by

$$\begin{aligned} Y_1 &= S^1 = X(S^1),\\ Y_2 &= S^1 \cup_\eta e^3 = X(\eta)_1,\\ Y_3 &= S^1 \cup_{\eta^2} e^4 = X(\eta^2)_1,\\ Y_4 &= S^2 = X(S^2),\\ Y_5 &= S^3 = X(S^3),\\ Y^v_1 &= X(\eta^2 v\eta^2)^1_0,\\ Y^v_2 &= X(\eta v\eta^2)^1_0,\\ Y^v_3 &= X(v\eta^2)^0_0. \end{aligned} \tag{3.3}$$

These spaces form the set $X(\mathscr{L}_4 - \mathscr{L}'_4)$; they are given by the following graphs:



Here the numbers d_i, d^v_j and the corresponding free abelian groups V_i and U^v_j are chosen in (3.5) and (3.8) below. Up to a suspension shift the space Y_i corresponds to the space X_i in 4.4 [4] for $i = 1, 2, \dots, 5$. In (3.5) (2) below we define the space Z_3 as an enlargement of the space $d_3 Y_3$ so that in this paper Z_3 replaces $d_3 Y_3$ in [4]. If $Z_3 = d_3 Y_3$ then the computation below coincides with the computation in [4].

3.4. Proposition. *$\Gamma_4 X(g) = 0$ for $g \in \mathscr{L}'_4$ and*

$$\Gamma_4(Y_1) = \mathbb{Z}/24,$$
$$\Gamma_4(Y_4) = \Gamma_4(Y_5) = \mathbb{Z}/2,$$
$$\Gamma_4(Y_2) = \Gamma_4(Y_3) = \Gamma_4(Y_j^v) = \mathbb{Z}/12.$$

Proof. We describe three examples. Consider $g = (\eta)_0 \in \mathscr{L}'_4$ so that $X(\eta)_0 = S^0 \cup_\eta e^2$. Then we have the stable cofiber sequence

$$S^1 \xrightarrow{\eta} S^0 \to X(\eta)_0 \to S^2 \xrightarrow{\eta} S^1,$$

which induces the exact sequence

$$\pi_4 S^0 \to \pi_4 X(\eta)_0 \to \pi_4 S^2 \xrightarrow{\eta_*} \pi_4 S^1.$$

Here $\pi_4 S^0 = 0$ by [14] and η_* is injective by [14]. Hence $\pi_4 X(\eta)_0 = 0$. In the same way we see $\pi_4 X(\eta^2)_0 = 0$. Moreover consider $g = (v\eta)_0^0 \in \mathscr{L}'_4$. Then we have

$$\Gamma_4 X(v\eta)_0^0 = \text{image}\{\pi_4(S^0 \vee S^2) = \pi_4 S^2 \to \pi_4 X(v\eta)_0^0\},$$

where $i_2 : S^2 \to X(v\eta)_0^0$ satisfies $i_0 v + i_2 \eta = 0$ by definition of $X(v\eta)_0^0$ in (1.3). Hence we get $i_2\eta^2 = i_0 v\eta = 0$ since $v\eta = 0$ by [14]. Hence $\Gamma_4 X(v\eta)_0^0 = 0$. Moreover, we use similar arguments for the computation of $\Gamma_4 X(g)$ with $g \notin \mathscr{L}'_4$. Here we need the relation $12(v + \alpha) = \eta\eta\eta$ in $\pi_4 S^1 = \mathbb{Z}/24$; compare [14]. For example we have the cofiber sequence

$$S^2 \xrightarrow{\eta} S^1 \to X(\eta)_1,$$

inducing the exact sequence

$$\pi_4 S^2 \to \pi_4 S^1 \to \pi_4 X(\eta)_1$$

so that $\Gamma_4 X(\eta)_1 = \text{cokernel}(\eta_* : \pi_4 S^2 \to \pi_4 S^1) = \mathbb{Z}/12$ since $\pi_4 S^2 = \mathbb{Z}/2$ is generated by η^2 and $\pi_4 S^1 = \mathbb{Z}/24$ is generated by $v + \alpha$. We point out that the generator of $\mathbb{Z}/12$ in the proposition is the composite $i_1(v + \alpha)$ where i_1 is the inclusion of S^1 into Y_2, Y_3 or Y_j^v. Q.E.D.

Since X^4 is a one-point union of spaces $X(g)$ with $g \in \mathscr{L}_4$ we can write

$$X^4 = L \vee \tilde{X}^4. \tag{3.5}$$

Here L is a one-point union of spaces $X(g)$ with $g \in \mathscr{L}'_4$ and $\tilde{X}^4$ is a one-point union of spaces in (3.3), that is,

$$\tilde{X}^4 = d_1 Y_1 \vee d_2 Y_2 \vee Z_3 \vee d_4 Y_4 \vee d_5 Y_5, \tag{7}$$

where Z_3 is a one-point union of spaces Y_3, Y_1^v, Y_2^v, Y_3^v namely

$$Z_3 = d_3 Y_3 \vee \bigvee_{v \in \{1,\ldots,6\}} (d_1^v Y_1^v \vee d_2^v Y_2^v \vee d_3^v Y_3^v). \tag{8}$$

Here d_i and d_j^v are numbers $\geqslant 0$. Using (3.4) we see that $\Gamma_4(L) = 0$ so that the attaching map (3.2) factors through the inclusion $\tilde{X}^4 \subset X^4$. Hence we get the following result which simplifies the proof of the decomposition theorem (2.5) a lot.

3.6. Proposition. *For X in $\mathbf{F}^5$ there is a homotopy equivalence $X \simeq L \vee \tilde{X}$. Here L is a one-point union of spaces $X(g)$ with $g \in \mathscr{L}'_4$ and $\tilde{X}$ is the cofiber of a map $\tilde{f}: c_5 S^4 \to \tilde{X}^4$ with $\tilde{X}^4$ as in* (3.5).

Hence in order to find all indecomposable spaces in $\mathbf{F}^5$ we only have to consider decompositions of spaces $\tilde{X}$ as in (3.6). We therefore assume in the following that $L = *$ so that $X = \tilde{X}$ satisfies $X^4 = \tilde{X}^4$ with $\tilde{X}^4$ as in (3.5) and $f = \tilde{f}$. In this case we say that X is *special.*

3.7. Proposition. *The space $Z = d_1 Y_1 \vee d_2 Y_2 \vee Z_3$ in* (3.5) *admits a surjection*

$$H_1(Z) \otimes \mathbb{Z}/24 \to \Gamma_4(Z),$$

which is natural in Z.

This is an easy consequence of (3.4) and the definition of Z. The complete analysis of the naturality properties of $\Gamma_4(X)$ for $X \in \mathbf{F}^4$ can be obtained by the homotopy operation spectral sequence of Baues–Goerss [6]. This shows (3.7) since for the space Z in (3.7) the composite

$$\pi_1(Z) \xrightarrow{(\eta\eta)^*} \pi_3(Z) \to (\pi_3(Z)/v^* \pi_0(Z)) \otimes \mathbb{Z}/2$$

is surjective.

We associate with a special space X the following free abelian groups; see (3.5):

$$\begin{aligned} &V_i = \mathbb{Z}^{d_i}, \\ &V = \bigoplus_{i=1}^{5} V_i, \\ &U_j^v = \mathbb{Z}^{d_j^v}, \quad U_j = \bigoplus_{v=1}^{6} U_j^v, \quad U^v = \bigoplus_{j=1}^{3} U_j^v, \\ &U = \bigoplus_{j=1}^{3} \bigoplus_{v=1}^{6} U_j^v = \bigoplus_{j=1}^{3} U_j = \bigoplus_{v=1}^{6} U^v. \end{aligned} \tag{3.8}$$

Then we obtain the homology groups $H_i = H_i(X)$ of the special space X by the formulas

$$\begin{aligned} H_0 &= U, \\ H_1 &= V_1 \oplus V_2 \oplus V_3 \oplus U, \\ H_2 &= V_4 \oplus U_2, \\ H_3 &= V_2 \oplus V_5 \oplus U_1, \\ H_4 &= V_3 \oplus U, \\ H_5 &= \mathbb{Z}^{c_5}. \end{aligned} \tag{9}$$

These formulas are determined by the basis elements in homology and have no functorial meaning. We also obtain for $\Gamma_4 = \Gamma_4(X^4)$ the formula

$$\Gamma_4 = V_1 \otimes \mathbb{Z}/24 \oplus (V_2 \oplus V_3 \oplus U) \otimes \mathbb{Z}/12 \oplus (V_4 \oplus V_5) \otimes \mathbb{Z}/2 \tag{10}$$

as a consequence of (3.4). This formula again is not natural.

3.9. Definition. We say that an automorphism $\varphi : \Gamma_4(X^4) \cong \Gamma_4(X^4)$ is *realizable* if there exists a homotopy equivalence $\alpha : X^4 \simeq X^4$ which induces φ; that is $\Gamma_4(\alpha) = \varphi$.

3.10. Sketch of proof. Let X be special. In order to prove the decomposition theorem (2.5) we have to find a purely algebraic characterization of all realizable automorphisms φ of $\Gamma_4(X^4)$. Of course $\alpha : X^4 \simeq X^4$ with $\Gamma_4(\alpha) = \varphi$ is not uniquely determined by φ and also the induced automorphism $H_*\alpha$ of the homology H_*X is not determined by φ. In order to characterize all realizable automorphisms φ of $\Gamma_4(X^4)$ as in (3.11) below we proceed as follows. Since X is special we know that X^4 has a decomposition as in (2.5) so that X^4 is a one-point union of spaces $X(g)$ with $g \in \mathscr{L}_4 - \mathscr{L}'_4$ as in (3.5) (1). We determine in (4.10) the algebraic maps

$$n^g_{g'} : H_* X(g) \to H_* X(g') \quad \text{(realizable)} \tag{11}$$

with $g, g' \in \mathscr{L}_4 - \mathscr{L}'_4$ for which there exists a map

$$\alpha^g_{g'} : X(g) \to X(g') \tag{12}$$

inducing $n^g_{g'}$ in homology. Hence each algebraic automorphism

$$n_* : H_*(X) \cong H_*(X) \tag{13}$$

which has coordinates as in (11) is in fact realizable by a homotopy equivalence $\alpha : X \simeq X$ with $H_*(\alpha) = n_*$, where α of course has coordinates in (12). We then describe in (4.5) the properties of the automorphism

$$\Gamma_4(\alpha) : \Gamma_4(X^4) \cong \Gamma_4(X^4), \tag{14}$$

which has coordinates $\Gamma_4(\alpha^g_{g'})$; compare (4.3) and (4.4). Since we are only interested in the realizability of an automorphism φ of $\Gamma_4(X^4)$ we proceed as follows. We show that for each realizable automorphism φ there exists an algebraic "proper automorphism" M which induces φ; see (4.6). Moreover given a proper automorphism M as defined in (3.12) we show that M induces an automorphism φ of $\Gamma_4(X^4)$ which is realizable; see (4.7). Hence we have an algebraic characterization of realizable automorphisms: An automorphism φ is realizable if and only if φ is induced by a proper automorphism M, see (3.11).

Remark. Since we are going to use integral representation theory associated to integral matrices in order to prove the decomposition result (2.5) we are forced to look for matrix data as in the definition of proper automorphism in (3.12) below. Such matrix data are canonical consequences of the decomposition of X^4 as a one-point union of spaces $X(g)$ with $g \in \mathscr{L}_4 - \mathscr{L}'_4$ as in (3.10). The definition of proper automorphism is very technical. In Section 4 we describe the topological meaning of this definition. It is, however, inevitable for us to separate the purely algebraic definition of a proper automorphism in (3.12) from the topology since this definition is the basis of the algebraic computations in Section 5. The proof of (3.11) below requires a lot of book keeping. Theorem (3.11) turns the topological classification problem into an algebraic classification problem.

3.11. Theorem. *An automorphism φ of $\Gamma_4(X^4)$ is realizable if and only if there exists a proper automorphism M of the free abelian group*

$$W = V_1 \oplus V_2 \oplus (V_3 \oplus U) \oplus V_4 \oplus V_5$$

for which the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, "M"] \arrow[d, "p"'] & W \arrow[d, "p"] \\ \Gamma_4(X^4) \arrow[r, "\varphi"] & \Gamma_4(X^4) \end{tikzcd}$$

Here p is the quotient map given by (3.8) (10).

3.12. Definition. Let $W_3 = V_3 \oplus U$ so that

$$W = V_1 \oplus V_2 \oplus W_3 \oplus V_4 \oplus V_5$$

is a direct sum of 5 summands. An automorphism M of the free abelian group W is *proper* if and only if M is given by a 5×5 matrix of the form

$$M = \begin{pmatrix} a_{11} & 2a_{12} & 2a_{13} & 12a_{14} & 12a_{15} \\ a_{21} & a_{22} & a_{23} & 12a_{24} & 6a_{25} \\ a_{31} & 2a_{32} & a_{33} & 12a_{34} & 12a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix} \tag{15}$$

with the following properties. The coordinates of the matrix M correspond to the direct sum decomposition of W above, for example $a_{15} \in \mathrm{Hom}(V_5, V_1)$, $a_{23} \in \mathrm{Hom}(W_3, V_2)$, $a_{33} \in \mathrm{Hom}(W_3, W_3)$. The submatrix

$$\mathscr{H}_1 = \begin{pmatrix} a_{11} & 2a_{12} & 2a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 2a_{32} & a_{33} \end{pmatrix} \tag{16}$$

of M is an automorphism of $V_1 \oplus V_2 \oplus W_3$. Recall that $W_3 = V_3 \oplus U$ where $U = U^1 \oplus \cdots \oplus U^6$ as in (3.8). We say that for

$$F, G \in \mathrm{Hom}(U, U)$$

the homomorphism F is $\mathbb{Z}/12$-*related* to the homomorphism G if the coordinates $F_{vw}, G_{vw} \in \mathrm{Hom}(U^w, U^v)$ of F and G satisfy for $v, w \in \{1, \ldots, 6\}$ the equation

$$(w \cdot F_{vw}) \otimes \mathbb{Z}/12 = (v \cdot G_{vw}) \otimes \mathbb{Z}/12. \tag{17}$$

We require that there exist automorphisms

$$\begin{aligned} &\mathscr{H}_0 \in \mathrm{Aut}(U) = \mathrm{Aut}(U_1 \oplus U_2 \oplus U_3), \\ &\mathscr{H}_4 \in \mathrm{Aut}(W_3) = \mathrm{Aut}(V_3 \oplus U), \end{aligned} \tag{18}$$

as follows. With respect to the direct sum decomposition $U = U_1 \oplus U_2 \oplus U_3$ in (3.8) the automorphism $\mathscr{H}_0$ is given by a matrix of the form

$$\mathscr{H}_0 = \begin{pmatrix} b_{11} & 2b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 2b_{31} & 2b_{32} & b_{33} \end{pmatrix}. \tag{19}$$

Moreover, with respect to the direct sum decomposition $W_3 = V_3 \oplus U$ the automorphism $\mathscr{H}_4$ is given by a matrix

$$\mathscr{H}_4 = \begin{pmatrix} c_{33} & c_{3U} \\ c_{U3} & c_{UU} \end{pmatrix}. \tag{20}$$

We require that

$$\mathscr{H}_0 \text{ is } \mathbb{Z}/12\text{-related to } c_{UU} \in \mathrm{Hom}(U, U) \tag{21}$$

and $c_{U3} \in \mathrm{Hom}(V_3, U)$ has coordinates $c_{U^v 3} \in \mathrm{Hom}(V_3, U^v)$ with $v \in \{1, \ldots, 6\}$ which satisfy the equation

$$(v \cdot c_{U^v 3}) \otimes \mathbb{Z}/12 = 0. \tag{22}$$

Finally, we require that $\mathscr{H}_4$ in (20) satisfies

$$\mathscr{H}_4 \otimes \mathbb{Z}/2 = a_{33} \otimes \mathbb{Z}/2, \tag{23}$$

where $a_{33} \in \mathrm{Hom}(W_3, W_3)$ is the coordinate of $\mathscr{H}_1$ in (16) above.

We point out that by (3.8) (9) we have the homology groups

$$\begin{aligned} H_0 &= U, \\ H_1 &= V_1 \oplus V_2 \oplus W_3, \\ H_4 &= W_3 = V_3 \oplus U, \end{aligned} \tag{3.13}$$

so that the automorphisms $\mathscr{H}_0$, $\mathscr{H}_1$, $\mathscr{H}_4$ in (3.12) are automorphisms of H_0, H_1 and H_4, respectively. Clearly, a homotopy equivalence $X^4 \simeq X^4$ induces such automorphisms $\mathscr{H}_0$, $\mathscr{H}_1$, $\mathscr{H}_4$ of homology and in addition automorphisms of H_2 and H_3 in (3.8) (9) which we obtain as follows.

3.14. Definition. Given M, $\mathscr{H}_0$, $\mathscr{H}_1$, $\mathscr{H}_4$ as in (3.12) we choose automorphisms $\mathscr{H}_2$ and $\mathscr{H}_3$ as follows. Let $\mathscr{H}_2$ be an automorphism of $H_2 = V_4 \oplus U_2$ which is given by a matrix

$$\mathscr{H}_2 = \begin{pmatrix} d_{44} & d_{42} \\ 2d_{24} & d_{22} \end{pmatrix},$$

satisfying Eqs. (24) and (25).

$$d_{44} \otimes \mathbb{Z}/2 = a_{44} \otimes \mathbb{Z}/2, \tag{24}$$

$$d_{22} \otimes \mathbb{Z}/2 = b_{22} \otimes \mathbb{Z}/2. \tag{25}$$

Here a_{44} and b_{22} are coordinates of M and $\mathscr{H}_0$, respectively. The properties of M and $\mathscr{H}_0$ readily show that $a_{44} \otimes \mathbb{Z}/2$ and $b_{22} \otimes \mathbb{Z}/2$ are automorphisms. Hence we can choose $\mathscr{H}_2$ since $GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/2)$ is surjective for all n; choose for example $d_{42} = 0$.

Moreover, let $\mathscr{H}_3$ be an automorphism of $H_3 = V_5 \oplus V_2 \oplus U_1$ which is given by a matrix

$$\mathscr{H}_3 = \begin{pmatrix} e_{55} & e_{52} & e_{51} \\ 2e_{25} & e_{22} & 2e_{21} \\ 2e_{15} & e_{12} & e_{11} \end{pmatrix},$$

satisfying Eqs. (26) … [29].

$$e_{55} \otimes \mathbb{Z}/2 = a_{55} \otimes \mathbb{Z}/2, \tag{26}$$

$$e_{22} \otimes \mathbb{Z}/2 = a_{22} \otimes \mathbb{Z}/2, \tag{27}$$

$$e_{11} \otimes \mathbb{Z}/2 = b_{11} \otimes \mathbb{Z}/2, \tag{28}$$

$$e_{25} \otimes \mathbb{Z}/2 = a_{25} \otimes \mathbb{Z}/2. \tag{29}$$

Here again a_{55}, a_{22}, a_{25} are coordinates of M in (3.12) (15) and b_{11} is a coordinate of $\mathscr{H}_0$ in (3.12) (19). Since a_{55} and $a_{22} \otimes \mathbb{Z}/2$ and $b_{11} \otimes \mathbb{Z}/2$ are automorphisms it is possible to choose $\mathscr{H}_3$. For example take $e_{52} = e_{51} = e_{21} = 0$ and use the surjection $GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/2)$.

Eq. (29) above corresponds to the equation in 4.6 (13) of [4].

4. Proof of Theorem (3.11)

Let $X^4 = \tilde{X}^4$ be a one-point union as in (3.5) (7), that is,

$$\begin{aligned} X^4 &= Z \vee d_4 Y_4 \vee d_5 Y_5, \\ Z &= d_1 Y_1 \vee d_2 Y_2 \vee Z_3, \\ Z_3 &= d_3 Y_3 \vee \bigvee_{v \in \{1,\dots,6\}} (d_1^v Y_1^v \vee d_2^v Y_2^v \vee d_3^v Y_3^v). \end{aligned} \tag{4.1}$$

Here Y_i and Y_j^v are given by (3.3). With respect to decomposition (4.1) a map $\alpha: X^4 \to X^4$ is given by a matrix

$$\alpha = \begin{pmatrix} \alpha_{ZZ} & \alpha_{Z4} & \alpha_{Z5} \\ \alpha_{4Z} & \alpha_{44} & \alpha_{45} \\ \alpha_{5Z} & 0 & \alpha_{55} \end{pmatrix} \tag{4.2}$$

with $\alpha_{Z5}: d_5 Y_5 \to Z$, etc. We clearly have $\alpha_{54} = 0$ since there are no essential maps from $Y_4 = S^2$ to $Y_5 = S^3$. Moreover α_{44} and α_{55} are determined by $H_2(\alpha)$ and $H_3(\alpha)$, respectively, and α_{45} is given by a homomorphism $qa_{45}: V_5 \to V_4 \to V_4 \otimes \mathbb{Z}/2$ where q is the quotient map. Clearly, a_{45} is not well defined by α_{45} and α_{44} and α_{55} need not to be automorphisms.

4.3. Proposition. *The maps α_{4Z} and α_{5Z} induce the trivial homomorphism on Γ_4, that is $\Gamma_4(\alpha_{4Z}) = \Gamma_4(\alpha_{5Z}) = 0$. Moreover, α_{Z4} has a coordinate $\alpha_{Z_3 4}: d_4 Y_4 \to Z_3$ and α_{Z5} has a coordinate $\alpha_{Z_3 5}: d_5 Y_5 \to Z_3$ with $\Gamma_4(\alpha_{Z_3 4}) = \Gamma_4(\alpha_{Z_3 5}) = 0$.*

Proof. We obtain $\Gamma_4(\alpha_{4Z}) = \Gamma_4(\alpha_{5Z}) = 0$ easily from (3.7) and (3.4) since composites $S^1 \to Z \to S^2, S^3$ are trivial. Moreover $\Gamma_4(\alpha_{Z_3 4}) = \Gamma_4(\alpha_{Z_3 5}) = 0$ is a consequence of the fact that $\Gamma_4 S^2$ and $\Gamma_4 S^3$ are generated by η^2 and η, respectively, and that $\eta\eta\eta$ is trivial in $\Gamma_4 Y_3$ and $\Gamma_4 Y_j^v$ by (3.4). Q.E.D.

Moreover we deduce from (3.7):

4.4. Proposition. *$\Gamma_4(\alpha_{ZZ}) = H_1(\alpha_{ZZ})_*$ is induced by $H_1(\alpha_{ZZ})$ and with respect to the decomposition $Z = d_1 Y_1 \vee d_2 Y_2 \vee Z_3$ the automorphism $H_1(\alpha_{ZZ})$ is given by a matrix of the form*

$$H_1(\alpha) = \begin{pmatrix} a_{11} & 2a_{12} & 2a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 2a_{32} & a_{33} \end{pmatrix}.$$

Proof. One readily checks that each realizable map $H_1 X(\eta)_1 \to H_1(S^1)$ is divisible by 2 so that we obtain $2a_{12}$. Similarly one readily checks for $\tilde{Y} = Y_3, Y_j^v$ that realizable maps $H_1 X(\eta)_1 \to H_1(\tilde{Y})$ and $H_1(\tilde{Y}) \to H_1(S^1)$ are divisible by 2. Hence we obtain $2a_{32}$ and $2a_{13}$. These facts are easy consequences of the attaching maps. Q.E.D.

Using (4.4) and (4.3) we see that with respect to the decomposition

$$\Gamma_4(X^4) = \Gamma_4(d_1 Y_1) \oplus \Gamma_4(d_2 Y_2) \oplus \Gamma_4(Z_3) \oplus \Gamma_4(d_4 Y_4) \oplus \Gamma_4(d_5 Y_5),$$

the homomorphism $\Gamma_4 \alpha$ is given by a matrix of the following form:

$$\Gamma_4 \alpha = \begin{pmatrix} \cdot & \cdot & \cdot & 12a_{14} & 12a_{15} \\ \cdot & \cdot & \cdot & 0 & 6a_{25} \\ \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & \alpha_{44} & a_{45} \\ 0 & 0 & 0 & 0 & \alpha_{55} \end{pmatrix}. \tag{4.5}$$

Here the dotted 3×3-matrix is induced by $H_1(\alpha)$ in (4.4). The coordinates a_{14}, a_{15} and a_{25} are obtained as follows. The map $\alpha_{15}: d_5 Y_5 \to d_1 Y_1$ is given by a homomorphism $a_{15} \otimes \mathbb{Z}/2 \in \mathrm{Hom}(V_5, V_1 \otimes \mathbb{Z}/2)$ since $Y_5 = S^3$ and $Y_1 = S^1$. Moreover, α_{15} induces on Γ_4 the commutative diagram

$$\begin{tikzcd} \Gamma_4(d_5Y_5) \arrow[r, "(\alpha_{15})_*"] \arrow[d, equal] & \Gamma_4(d_1Y_1) \arrow[d, equal] \\ V_5\otimes\mathbb{Z}/2 \arrow[r, "12a_{15}"] & V_1\otimes\mathbb{Z}/24 \end{tikzcd}$$

A similar argument holds for the coordinate a_{14}. Moreover, $\alpha_{25}: d_5Y_5 \to d_2Y_2$ is given by a homomorphism $2a_{25} \in \mathrm{Hom}(V_5, V_2)$ since $Y_2 = S^1 \cup_\eta e^3$ and $Y_5 = S^3$ and $a_{25} = H_3(\alpha_{25})$. Moreover α_{25} induces in Γ_4 the commutative diagram

$$\begin{tikzcd} \Gamma_4(d_5Y_5) \arrow[r, "(\alpha_{25})_*"] \arrow[d, equal] & \Gamma_4(d_2Y_2) \arrow[d, equal] \\ V_5\otimes\mathbb{Z}/2 \arrow[r, "6a_{25}"] & V_2\otimes\mathbb{Z}/12 \end{tikzcd}$$

Here we use the Toda bracket $2v \in \langle \eta, e, \eta \rangle$, compare also Unsöld [15,16].

If the map α is a homotopy equivalence then clearly $H_1(\alpha)$ is an automorphism and by the form (4.5) of the automorphism $\Gamma_4\alpha$ we see that also $\alpha_{44}\otimes\mathbb{Z}/2$ and $\alpha_{55}\otimes\mathbb{Z}/2$ are automorphisms. Using the surjection $GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/2)$ we can choose automorphisms a_{44} and a_{55} over $\mathbb{Z}$ with $a_{44}\otimes\mathbb{Z}/2 = \alpha_{44}\otimes\mathbb{Z}/2$ and $a_{55}\otimes\mathbb{Z}/2 = \alpha_{55}\otimes\mathbb{Z}/2$. Moreover we choose arbitrary morphisms a_{24}, a_{34}, a_{35} as in (3.12). Hence we obtain by such choices and by $H_1(\alpha)$ and $\Gamma_4\alpha$ above a matrix M as in (3.12) (15) which we call a matrix *associated* to the homotopy equivalence α above. By definition of M diagram (3.11) with $\varphi = \Gamma_4(\alpha)$ commutes and M is an automorphism.

In order to prove (3.11) we have to show the following two lemmas.

4.6. Lemma. *Let α be a homotopy equivalence as above. Then $\mathscr{H}_1 = H_1(\alpha), \mathscr{H}_0 = H_0(\alpha)$, and $\mathscr{H}_4 = H_4(\alpha)$ together with a matrix M associated to α as above have properties as described in* (3.12). *In particular, a matrix M associated to α is a proper automorphism. Hence for any realizable automorphism φ of $\Gamma_4(\alpha)$ there exists a proper automorphism M for which diagram* (3.11) *commutes.*

4.7. Lemma. *Let M be a proper automorphism as in* (3.12) *so that we can choose $\mathscr{H}_0, \mathscr{H}_1, \mathscr{H}_2, \mathscr{H}_3, \mathscr{H}_4$ as in* (3.12) *and* (3.14). *Then there exists a map $\alpha: X^4 \to X^4$ with $H_i(\alpha) = \mathscr{H}_i$ such that diagram* (3.11) *with $\varphi = \Gamma_4(\alpha)$ commutes. Here α is a homotopy equivalence since $\mathscr{H}_i$ are automorphisms. Hence, each proper automorphism M induces via the diagram in* (3.11) *an automorphism φ of $\Gamma_4(X^4)$ which is realizable.*

It is clear that (4.6) and (4.7) yield a proof of (3.11). In order to prove (4.6) and (4.7) we repeat the classification theorem of Unsöld [15,16]. He defines the following algebraic category $\mathbf{SF}^4$.

4.8. Definition. Objects in $\mathbf{SF}^4$ are tuples of abelian groups

$$\mathscr{H} = (H_0, H_1, H_2, H_3, H_4, \pi_1, \pi_2) \in \mathbf{Ab}^7$$

where H_i with $i \in \{0, \ldots, 4\}$ is finitely generated and free abelian together with the following diagrams (30)–(33) in **Ab**:

(30) An exact sequence

$$H_3 \to \pi_1 \otimes \mathbb{Z}/2 \to \pi_2 \to H_2 \xrightarrow{b} H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta^1} \pi_1 \to H_1 \to 0.$$

(31) Let $P = \ker(H_0 \xrightarrow{q} H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta^1} \pi_1)$ where q is the quotient map. Then

$$\begin{tikzcd} 0 \arrow[r] & H_0 \arrow[r, "2"] \arrow[d, "q\eta^1 q"'] & P \arrow[r] \arrow[d, "T"] & \ker(\eta^1) \arrow[r] \arrow[d, "\Omega"] & 0 \\ & \pi_1 \otimes \mathbb{Z}/2 \arrow[r] & \pi_2 \otimes \mathbb{Z}/2 \arrow[r] & \ker(b) \otimes \mathbb{Z}/2 & \end{tikzcd}$$

commutes where $q\eta^1 q$ is given by (30) and where Ω is determined by the extension

$$0 \to \ker(b) \to H_2 \to \ker(\eta^1) \to 0$$

given by (30). The top row of the diagram is short exact.

(32) Moreover, for the abelian group

$$\Gamma_3 = (H_0 \otimes \mathbb{Z}/24 \oplus \pi_2 \otimes \mathbb{Z}/2)/\{(\xi \otimes 6, T(\xi)); \xi \in P \subset H_0\}$$

defined by T in (31) a homomorphism

$$b_4 : H_4 \to \Gamma_3$$

is given.

A morphism between such objects in $\mathbf{SF}^4$ is a tuple of homomorphisms $\mathscr{H} \to \mathscr{H}'$ in $\mathbf{Ab}^7$ which is compatible with all arrows in the diagrams (30)–(33). Clearly, $\mathbf{SF}^4$ is an additive category with the direct sum of objects given by the direct sum of abelian groups and morphisms.

In [15,16] one finds the proof of the following result.

4.9. Theorem. *There is an additive functor* $\lambda : \mathbf{F}^4 \to \mathbf{SF}^4$ *which is full and representative and which reflects isomorphisms.*

The functor carries a space X to the certain exact sequence of Whitehead [18] of X together with the secondary homotopy operation T which was introduced by Unsöld.

Theorem (4.9) allows the computation of all realizable homology homomorphisms

$$n_*: H_*(Y) \to H_*(Y') \tag{4.10}$$

with $Y, Y' \in X(\mathscr{L}_4 - \mathscr{L}'_4)$; see (3.3). For this we use the fact that the functor λ in (4.9) is full. One readily checks that $H_i(Y)$ is either 0 or $\mathbb{Z}$ so that n_* is given by $n_i \in \mathbb{Z}$ with $i \in \{0, \dots, 4\}$. Below we describe the non-trivial n_i for which $n_* = (n_0, n_1, n_2, n_3, n_4)$ is realizable by a map $Y \to Y'$. For example, if Y or Y' are spheres with $Y \neq Y'$ one can check that there are only the following possibilities (33)–(37):

$$Y_1 \xrightarrow{n_1} Y_3 \xrightarrow{n_1 \equiv 0\ (2)} Y_1, \tag{33}$$

$$Y_1 \xrightarrow{n_1} Y_2 \xrightarrow{n_1 \equiv 0\ (2)} Y_1, \quad Y_5 \xrightarrow{n_3 \equiv 0\ (2)} Y_2 \xrightarrow{n_3} Y_5, \tag{34}$$

$$Y_1 \xrightarrow{n_1} Y_1^v \xrightarrow{n_1 \equiv 0\ (2)} Y_1, \quad Y_5 \xrightarrow{n_3 \equiv 0\ (2)} Y_1^v \xrightarrow{n_3} Y_5, \tag{35}$$

$$Y_1 \xrightarrow{n_1} Y_2^v \xrightarrow{n_1 \equiv 0\ (2)} Y_1, \quad Y_4 \xrightarrow{n_2 \equiv 0\ (2)} Y_2^v \xrightarrow{n_2} Y_4, \tag{36}$$

$$Y_1 \xrightarrow{n_1} Y_3^v \xrightarrow{n_1 \equiv 0\ (2)} Y_1. \tag{37}$$

Moreover if Y or Y' are Hopf elements we get the following possibilities (38)–(46).

$$Y_2 \xrightarrow{n_1 \equiv n_2\ (2)} Y_2, \tag{38}$$

$$Y_3 \xrightarrow{n_1 \equiv n_4\ (2)} Y_3, \tag{39}$$

$$Y_2 \xrightarrow{n_1 \equiv 0\ (2)} Y_3 \xrightarrow{n_1} Y_2, \tag{40}$$

$$Y_2 \xrightarrow[n_3]{n_1 \equiv 0\ (2)} Y_1^v \xrightarrow[n_3 \equiv 0\ (2)]{n_1} Y_2, \tag{41}$$

$$Y_3 \xrightarrow[n_4 v \equiv 0\ (12)]{n_4 \equiv n_1\ (2)} Y_1^v \xrightarrow{n_1 \equiv n_4\ (2)} Y_3, \tag{42}$$

$$Y_2 \xrightarrow{n_1 \equiv 0\ (2)} Y_2^v \xrightarrow{n_1} Y_2, \tag{43}$$

$$Y_3 \xrightarrow[n_4 v \equiv 0\ (12)]{n_1 \equiv n_4\ (2)} Y_2^v \xrightarrow{n_1 \equiv n_4\ (2)} Y_3, \tag{44}$$

$$Y_2 \xrightarrow{n_1 \equiv 0\ (2)} Y_3^v \xrightarrow{n_1} Y_2, \tag{45}$$

$$Y_3 \xrightarrow[n_4 v \equiv 0\ (12)]{n_1 \equiv n_4\ (2)} Y_3^v \xrightarrow{n_1 \equiv n_4\ (2)} Y_3. \tag{46}$$

Finally we get for $Y,\ Y' \in \{Y_1^v, Y_2^v, Y_3^v\}$ the following possibilities (47)–(55) with $v, w \in \{1, \ldots, 6\}$:

$$Y_1^w \xrightarrow[n_0 w \equiv n_4 v\ (12)]{n_0 \equiv n_3\ (2),\ n_1 \equiv n_4\ (2)} Y_1^v, \tag{47}$$

$$Y_1^w \xrightarrow[n_0 w \equiv n_4 v\ (12)]{n_4 \equiv n_1\ (2)} Y_2^v, \tag{48}$$

$$Y_1^w \xrightarrow[n_4 v \equiv n_0 w\ (12)]{n_0 \equiv 0\ (2),\ n_4 \equiv n_1\ (2)} Y_3^v, \tag{49}$$

$$Y_2^w \xrightarrow[n_0 w \equiv n_4 v\ (12)]{n_0 \equiv 0\ (2),\ n_1 \equiv n_4\ (2)} Y_1^v, \tag{50}$$

$$Y_2^w \xrightarrow[n_0 w \equiv n_4 v\ (12)]{n_2 \equiv n_0\ (2),\ n_1 \equiv n_4\ (2)} Y_2^v, \tag{51}$$

$$Y_2^w \xrightarrow[n_0 w \equiv n_4 v\ (12)]{n_0 \equiv 0\ (2),\ n_4 \equiv n_1\ (2)} Y_3^v, \tag{52}$$

$$Y_3^w \xrightarrow[n_0 w \equiv n_4 v\ (12)]{n_4 \equiv n_1\ (2)} Y_1^v, \tag{53}$$

$$Y_3^w \xrightarrow[n_0 w \equiv n_4 v\ (12)]{n_1 \equiv n_4\ (2)} Y_2^v, \tag{54}$$

$$Y_3^w \xrightarrow[n_0 w \equiv n_4 v\ (12)]{n_1 \equiv n_4\ (2)} Y_3^v. \tag{55}$$

Proof of (33)–(55). One can prove the conditions of realizability of $n_* = (n_1, \ldots, n_4)$ in (33)–(55) either directly by use of the cell structure of $Y, Y' \in X(\mathscr{L}_4 - \mathscr{L}'_4)$ or by use of Theorem (4.9). In order to apply (4.9) we point out that the objects in $\mathbf{SF}^4$ corresponding to Y_3, Y_1^v, Y_2^v, Y_3^v are given by the following list:

	$H_3 \longrightarrow$	$\pi_1\otimes\mathbb{Z}/2 \longrightarrow$	$\pi_2 \longrightarrow$	$H_2 \longrightarrow$	$H_0\otimes\mathbb{Z}/2 \longrightarrow$	$\pi_1 \twoheadrightarrow$	$H_1,$	H_0
Y_3	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	$\mathbb{Z}$	$\mathbb{Z}$	0
Y_1^v	$\mathbb{Z} \xrightarrow{(1,0)}$	$\mathbb{Z}/2\oplus\mathbb{Z}/2 \xrightarrow{(0,1)}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
Y_2^v	0	$\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
Y_3^v	0	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	0	0	$\mathbb{Z}$	$\mathbb{Z}$	0

	$H_0 \supset$	$P \xrightarrow{T}$	$\pi_2\otimes\mathbb{Z}/2,$	$H_4 \longrightarrow$	$\Gamma_3 \twoheadleftarrow$	$H_0\otimes\mathbb{Z}/24\oplus\pi_2\otimes\mathbb{Z}/2$
Y_3	0	0	$\mathbb{Z}/2$	$\mathbb{Z} \xrightarrow{1}$	$\mathbb{Z}/2 \longleftarrow$ y	$0\oplus\mathbb{Z}/2$ y
Y_1^v	$\mathbb{Z}$	$2\mathbb{Z} \xrightarrow{0}$	$\mathbb{Z}/2$	$\mathbb{Z} \xrightarrow{(v,1)}$	$\mathbb{Z}/12\oplus\mathbb{Z}/2 \longleftarrow$ (x,y)	$\mathbb{Z}/24\oplus\mathbb{Z}/2$ (x,y)
Y_2^v	$\mathbb{Z} =$	$\mathbb{Z} \xrightarrow{(0,1)}$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z} \xrightarrow{(v,1)}$	$\mathbb{Z}/12\oplus\mathbb{Z}/2 \longleftarrow$ $(x+6z,y)$	$\mathbb{Z}/24\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$ (x,y,z)
Y_3^v	$\mathbb{Z}$	$2\mathbb{Z} \xrightarrow{(1/2,0)}$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z} \xrightarrow{(v,1)}$	$\mathbb{Z}/24\oplus\mathbb{Z}/2 \longleftarrow$ $(x+12z,z)$	$\mathbb{Z}/24\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$ (x,y,z)

Using these objects of $\mathbf{SF}^4$ corresponding to Y_3, Y_1^v, Y_2^v, Y_3^v it is an easy but inevitable calculation to obtain the conditions on n_* in (23)–(55) above. For example if we consider maps $Y_2^w \to Y_1^v$ we obtain $n_* = (n_0, n_1, n_4)$ such that the following diagrams commute (these diagrams describe a morphism $\lambda(Y_2^w) \to \lambda(Y_1^v)$ in $\mathbf{SF}^4$):

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r, "{(1,0)}"] \arrow[d, "{(\alpha,n_1)}"] & \mathbb{Z}/2\oplus\mathbb{Z} \arrow[r] \arrow[d, "{(n_1,\beta)}"] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r, "0"] \arrow[d, "n_0"] & \mathbb{Z} \arrow[r] \arrow[d, "{(\alpha,n_1)}"] & \mathbb{Z}, \arrow[d, "n_1"] & \mathbb{Z} \arrow[d, "n_0"] \\
\mathbb{Z} \arrow[r, "{(1,0)}"] & \mathbb{Z}/2\oplus\mathbb{Z}/2 \arrow[r, "{(0,1)}"] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "{(1,0)}"] & \mathbb{Z}/2\oplus\mathbb{Z} \arrow[r] & \mathbb{Z}, & \mathbb{Z}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, equal] \arrow[d, "n_0"'] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/2\oplus\mathbb{Z}/2\ ; \arrow[d, "{(n_1,\beta)}"] \\
\mathbb{Z} \supset & 2\mathbb{Z} \arrow[r, "0"] & \mathbb{Z}/2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Z} \arrow[r, "{(w,1)}"] \arrow[d, "n_4"'] & \mathbb{Z}/12\oplus\mathbb{Z}/2 \arrow[d, "n_0\oplus n_1"] & \mathbb{Z}/24\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2 \arrow[l] \arrow[d, "{n_0\oplus(n_1,0)}"'] \\
\mathbb{Z} \arrow[r, "{(v,1)}"] & \mathbb{Z}/12\oplus\mathbb{Z}/2 & \mathbb{Z}/24\oplus\mathbb{Z}/2 \arrow[l]
\end{tikzcd}$$

The first diagram shows $n_0 \equiv 0\ (34)$ and the last diagram shows $n_1 \equiv n_4\ (2)$ and $n_0 w \equiv n_4 v\ (12)$. If $n_* = (n_0, n_1, n_4)$ satisfies these conditions then α and $\beta = 0$ can be chosen such that the diagrams commute. Q.E.D.

Hence homology homomorphisms (4.10) are realizable if and only if the conditions (23)–(55) hold. Hence an automorphism n_* of $H_*(X)$ as in (3.10) (13) is realizable if and only if the coordinates of n_* satisfy the conditions in (23)–(55).

Proof of 4.6. We start with a homotopy equivalence α and we obtain the associated automorphism M as in (4.5) above. We have to check that M satisfies the conditions in (3.12). Clearly (3.12) (15), (16) are satisfied. Let $\mathscr{H}_0 = H_0(\alpha)$ and $\mathscr{H}_4 = H_4(\alpha)$ in (3.12) (18). Then (19), (21)–(23) are consequences of (4.10) (33)–(55). Q.E.D.

Proof of 4.7. Here we start with a proper automorphism M. We choose $\mathscr{H}_0, \mathscr{H}_1, \mathscr{H}_2, \mathscr{H}_3, \mathscr{H}_4$ associated to M as in (3.12) and (3.14). Then (4.10) (33)–(55) show that there is a homotopy equivalence $\alpha : X^4 \simeq X^4$ with $H_i(\alpha) = \mathscr{H}_i$ for $i = 0, 1, 2, 3, 4$. Here α has a form as in (4.2) where the coordinates α_{14}, α_{15}, and α_{45} have no effect in homology. Therefore $\alpha_{14}, \alpha_{15}, \alpha_{45}$ can be chosen appropriately in order to make diagram (3.11) with $\varphi = \Gamma_4(\alpha)$ commutative. Q.E.D.

5. Simplification of proper automorphisms

The condition describing a proper automorphism in (3.12) can be considerably simplified as follows.

5.1. Proposition. *An automorphism M of W is proper if and only if with respect to the decomposition*

$$W = V_1 \oplus V_2 \oplus V_3 \oplus U^1 \oplus U^2 \oplus U^3 \oplus U^4 \oplus U^5 \oplus U^6 \oplus V_4 \oplus V_5$$

the matrix M is a matrix of the form

$$\begin{pmatrix}
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 12 & 12 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 12 & 6 \\
1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 12 & 12 \\
1 & 2 & 2 & \bigstar & 2 & \bigstar & 2 & \bigstar & 2 & 12 & 12 \\
1 & 2 & 2 & 1 & \bigstar & 1 & 2 & 1 & \bigstar & 12 & 12 \\
1 & 2 & 2 & \bigstar & 2 & \bigstar & 2 & \bigstar & 2 & 12 & 12 \\
1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 12 & 12 \\
1 & 2 & 2 & \bigstar & 2 & \bigstar & 2 & \bigstar & 2 & 12 & 12 \\
1 & 2 & 2 & 1 & \bigstar & 1 & 2 & 1 & \bigstar & 12 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Here an integer m at some place of a matrix means that the corresponding block equals ma where a can be any integral matrix. Moreover, $\bigstar$ means that this block with respect to the decomposition

$$U^v = U_1^v \oplus U_2^v \oplus U_3^v$$

is a matrix of the form

$$\bigstar = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}.$$

Remark. If all U^i with $i \in \{1, \dots, 6\}$ are trivial then the square in the matrix of (5.1) shrinks to a 1×1-matrix and in this case we get the matrix problem for $\mathbf{F}^4$ which was solved in [4].

Proof. The necessity of this condition follows readily from the definition of a proper automorphism. To prove sufficiency consider the submatrix N of M corresponding to the direct summand $W_3 = V_3 \oplus U$. The submatrix N is indicated by the rectangle in the matrix M above. Using the notation $U^0 = V_3$ the matrix N has blocks $N_{ij} : U^j \to U^i$ given with respect to the decomposition

$$W_3 = U^0 \oplus U^1 \oplus U^2 \oplus U^3 \oplus U^4 \oplus U^5 \oplus U^6.$$

Since M is invertible also N is invertible modulo 2. Hence the following submatrices of N are also invertible modulo 2:

$$N_0 = \begin{pmatrix} N_{00} & N_{04} \\ N_{40} & N_{44} \end{pmatrix},$$

$$N_2 = \begin{pmatrix} N_{22} & N_{26} \\ N_{62} & N_{66} \end{pmatrix},$$

$$N_1 = \begin{pmatrix} N_{11} & N_{13} & N_{15} \\ N_{31} & N_{33} & N_{35} \\ N_{51} & N_{53} & N_{55} \end{pmatrix}.$$

Consider the ring R_0 of integral matrices of the form

$$B_0 = \begin{pmatrix} B_{00} & B_{04} \\ 3B_{40} & B_{44} \end{pmatrix},$$

where B_{vw} has the same size as N_{vw}. Then we get

$$R_0/2R_0 = \mathrm{Mat}(n, \mathbb{F}_2),$$

where n is the number of rows in N_0. As $GL(n, \mathbb{F}_2) = SL(n, \mathbb{F}_2)$ is generated by elementary matrices the natural homomorphism

$$R_0^\times \to GL(n, \mathbb{F}_2)$$

is surjective. In particular, there exists an invertible matrix $B_0 \in R_0$ such that $B_0 \equiv N_0 \pmod 2$. Moreover the matrix B_{44} is invertible modulo 3.

Next let R be the ring of all matrices of the form $\bigstar$. Then

$$R/3R = \mathrm{Mat}(r, \mathbb{F}_3).$$

The group $GL(r,\mathbb{F}_3)$ is generated by elementary matrices and the matrix $\mathrm{diag}(-1,1,\dots,1)$. Hence the natural homomorphism

$$R^\times \to GL(r,\mathbb{F}_3)$$

is also surjective; in particular, there exists an invertible matrix $A_{44} \in R$ such that $A_{44} \equiv B_{44} \pmod 3$.

Similar observations show that there exist invertible matrices

$$B_2 = \begin{pmatrix} B_{22} & 3B_{26} \\ B_{62} & B_{66} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} B_{11} & 3B_{13} & B_{15} \\ B_{31} & B_{33} & B_{35} \\ B_{51} & 3B_{53} & B_{55} \end{pmatrix},$$

where all blocks B_{wv} are of the form $\bigstar$ and for which in addition $B_2 \equiv N_2 \pmod 2$ and $B_1 \equiv N_1 \pmod 2$. Now put

$$B_1' = \begin{pmatrix} B_{11} & B_{13} & 5B_{15} \\ 3B_{31} & B_{33} & 3B_{35} \\ 5B_{51} & 5B_{53} & B_{55} \end{pmatrix},$$

so that

$$B_1' \equiv \begin{pmatrix} I & 0 & 0 \\ 0 & 3I & 0 \\ 0 & 0 & 5I \end{pmatrix} B_1 \begin{pmatrix} I & 0 & 0 \\ 0 & \frac{1}{3}I & 0 \\ 0 & 0 & 5I \end{pmatrix} \pmod{12}$$

and $\det B'_1 \equiv \det B_1 \equiv \pm 1 \pmod{12}$. Therefore just as above there is an invertible matrix

$$A_1 = \begin{pmatrix} A_{11} & A_{13} & 5A_{15} \\ 3A_{31} & A_{33} & 3A_{35} \\ 5A_{51} & 5A_{53} & A_{55} \end{pmatrix}$$

such that $A_1 \equiv B_1' \pmod{12}$. Moreover, this implies that all blocks A_{wv} have the form $\bigstar$.

On the other hand, as B_2 is invertible its conjugate

$$A_2 = \begin{pmatrix} I & 0 \\ 0 & 3I \end{pmatrix} B_2 \begin{pmatrix} I & 0 \\ 0 & \frac{1}{3}I \end{pmatrix} = \begin{pmatrix} B_{22} & B_{26} \\ 3B_{62} & B_{66} \end{pmatrix}$$

is also invertible. Put $A_{wv} = B_{wv}$ for $v,w \in \{2,6\}$.

We are now ready to define $\mathscr{H}_0$ and $\mathscr{H}_4$ associated to M such that the properties in (3.12) hold. This proves that M is proper and hence completes the proof of (5.1). Let $\mathscr{H}_4$ be the matrix

$$\mathscr{H}_4 = \left(\begin{array}{c|cccccc} B_{00} & B_{01} & B_{02} & B_{03} & B_{04} & B_{05} & B_{06} \\ \hline 0 & B_{11} & 0 & 3B_{13} & 0 & B_{15} & 0 \\ 0 & B_{21} & B_{22} & 3B_{23} & 0 & B_{25} & B_{26} \\ 0 & B_{31} & 0 & B_{33} & 0 & B_{35} & 0 \\ 3B_{40} & B_{41} & B_{42} & 3B_{43} & B_{44} & B_{45} & 3B_{46} \\ 0 & B_{51} & 0 & 3B_{53} & 0 & B_{55} & 0 \\ 0 & B_{61} & B_{62} & B_{63} & 0 & B_{65} & B_{66} \end{array}\right)$$

Here all blocks B_{wv} which have not yet been defined coincide with N_{wv}. Then $\mathscr{H}_4$ is invertible and $\mathscr{H}_4 \equiv N \,(\mathrm{mod}\, 2)$. Next let

$$\mathscr{H}_0 = \begin{pmatrix} A_{11} & 0 & A_{13} & 0 & 5A_{15} & 0 \\ 2A_{21} & A_{22} & 2A_{23} & 0 & 2A_{25} & A_{26} \\ 3A_{31} & 0 & A_{33} & 0 & 3A_{35} & 0 \\ 4A_{41} & 2A_{42} & 4A_{43} & A_{44} & 4A_{45} & 2A_{46} \\ 5A_{51} & 0 & 5A_{53} & 0 & A_{55} & 0 \\ 6A_{61} & 3A_{62} & 2A_{63} & 0 & 6A_{65} & A_{66} \end{pmatrix}$$

where all blocks A_{wv} which have not yet been defined coincide with B_{wv}. Then A is also invertible and, in fact, A is $\mathbb{Z}/12$-related to the lower right part of $\mathscr{H}_4$ corresponding to the direct summand U and indicated by the rectangle in $\mathscr{H}_4$ above. Hence this proves that M is proper; compare (3.12). Q.E.D.

6. Proof of the decomposition Theorem (2.5)

Let Λ be the ring of all integral (23×23)-matrices of form (5.1), whose rows and columns are numbered by the indexes $i = 1, \dots, 5$ and pairs $\binom{v}{j}$ with $v = 1, \dots, 6$; $j = 1, \dots, 3$. Let

$$\mathscr{U}_1 = \mathbb{Z}/24,$$

$$\mathscr{U}_2 = \mathscr{U}_3 = \mathscr{U}_{\binom{v}{j}} = \mathbb{Z}/12,$$

$$\mathscr{U}_4 = \mathscr{U}_5 = \mathbb{Z}/2.$$

Then the direct sum

$$\mathscr{U} = \bigoplus_{i=1}^{5} \mathscr{U}_i \oplus \bigoplus_{v=1}^{6} \bigoplus_{j=1}^{3} \mathscr{U}_{\binom{v}{j}}$$

is in the obvious way a Λ-$\mathbb{Z}$-bimodule denoted by $\mathscr{U} = {}_{\Lambda}\mathscr{U}_{\mathbb{Z}}$. Recall that a *matrix over* $\mathscr{U}$ is by definition (cf. [5]) an element of $P \otimes_{\Lambda} \mathscr{U} \otimes H^*$ where P and H are finitely generated right projective modules over Λ and $\mathbb{Z}$, respectively. It is more convenient to identify this tensor product with

$$\mathscr{U}(H, P) = \mathrm{Hom}(H, P \otimes_{\Lambda} \mathscr{U}).$$

Two matrices $u \in \mathscr{U}(H, P)$, $u' \in \mathscr{U}(H', P')$ are *isomorphic* if there are isomorphisms $\alpha : H \to H'$ and $\beta : P \to P'$ such that $\beta u = u' \alpha$.

Let $e_i = e_{ii}$ and $e_{\binom{v}{j}} = e_{\binom{v}{j}\binom{v}{j}}$ be matrix units and

$$P_i = e_i \Lambda, \quad P_{\binom{v}{j}} = e_{\binom{v}{j}} \Lambda.$$

Then P can be uniquely decomposed as

$$P = \left(\bigoplus_{i=1}^{5} V_i \otimes P_i \right) \oplus \bigoplus_{v=1}^{6} \bigoplus_{j=1}^{3} U_j^v \otimes P_{\binom{v}{j}}$$

for some free abelian groups V_i, U_j^v. Therefore,

$$P \otimes_{\Lambda} \mathscr{U} \cong \left(\bigoplus_{i=1}^{5} V_i \otimes \mathscr{U}_i \right) \oplus \bigoplus_{v=1}^{6} \bigoplus_{j=1}^{3} U_j^v \otimes \mathscr{U}_{\binom{v}{j}}.$$

This shows that isomorphism classes of matrices u above are in 1–1 correspondence with homotopy types in $\mathbf{F}^5$ (for this we use (3.11) and (5.1)).

We shall write the elements of $\mathscr{U}(H, P)$ as families of matrices (u_i, u_j^v) with u_i being of size $d_i \times c_5$ and u_j^v of size $d_j^u \times c_5$ if

$$H = \mathbb{Z}^{c_5},$$

$$V_i = \mathbb{Z}^{d_i},$$

$$U_j^v = \mathbb{Z}^{d_j^v},$$

as in (3.5) and (3.8). The entries of u_i, u_j^v are in the corresponding groups $\mathscr{U}_i$ and $\mathscr{U}_{\binom{v}{j}}$, respectively. The matrices $M \in \Lambda$ define the "admissible transformations" of rows in these matrices. For instance, as we have a_{21} in M we can add any multiple of a row of the matrix u_1 to any row of the matrix u_2. On the other hand, as we have $2a_{12}$ in M, we can add only even multiples of rows of u_2 to the rows of u_1, etc.

We point out that the ring Λ can be "Morita-reduced" in the following way. Denote by e_i $(i = 1, 2, 3, 4, 5)$ and e_j^v $(j = 1, 2, 3;\ v = 1, 2, 3, 4, 5, 6)$ the idempotent matrices from Λ having all zero entries except one, which equals 1 and is sitting on the diagonal place corresponding to the direct summand V_i or U_j^v, respectively. Put

$$P_i = e_i \Lambda,$$

$$P_j^v = e_j^v \Lambda.$$

These are projective Λ-modules. We get

$$\Lambda = \left(\bigoplus_{i} P_i \right) \oplus \left(\bigoplus_{j,v} P_j^v \right).$$

One can easily obtain the following isomorphisms with $j = 1,2,3$:

$$P_3 \cong P_j^4,$$
$$P_j^1 \cong P_j^3 \cong P_j^5,$$
$$P_j^2 \cong P_j^6.$$

Therefore the module

$$P = \left(\bigoplus_i P_i\right) \oplus \left(\bigoplus_j (P_j^3 \oplus P_j^6)\right)$$

is a pro-generator for Λ. Hence Λ is Morita equivalent to

$$\Lambda' = \mathrm{End}\,(\Lambda P).$$

Indeed, the ring Λ' consists of the matrices of the same form as in Λ for which rows and columns only correspond, however, to the indexes $i = 1,2,3,4,5$ and the pairs (v,j) with $v = 3,6$ and $j = 1,2,3$. Let

$$\mathscr{U}' = P \otimes_\Lambda \mathscr{U}.$$

This is a Λ'-$\mathbb{Z}$-bimodule (the restriction of the bimodule $\mathscr{U}$ to Λ). Hence the categories of $\mathscr{U}$-matrices and $\mathscr{U}'$-matrices are equivalent.

Moreover, $\mathscr{U}'$ is also a $\bar{\Lambda}$-$\bar{\mathbb{Z}}$-bimodule with $\bar{\mathbb{Z}} = \mathbb{Z}/24$ and $\bar{\Lambda} = \Lambda'/24$, which we denote by $\bar{\mathscr{U}} = {}_{\bar{\Lambda}}\mathscr{U}_{\bar{\mathbb{Z}}}$. The elements of $\mathscr{U}$ and $\bar{\mathscr{U}}$ are the same and also the matrices from $\mathscr{U}(P,H)$ coincide with those from $\bar{\mathscr{U}}(\bar{P},\bar{H})$ with $\bar{P} = P/24$ and $\bar{H} = H/24$, but non-isomorphic $\mathscr{U}$-matrices might be isomorphic as $\bar{\mathscr{U}}$-matrices.

Consider the 2-primary part $\tilde{\mathscr{U}}$ of $\bar{\mathscr{U}}$ and let $\tilde{\Lambda}$ be the ring of matrices with entries in $\mathbb{Z}/8$ satisfying the same conditions as matrices in Λ' above (i.e., in the corresponding congruences we replace (mod 24) by (mod 8), (mod 12) by (mod 4) and (mod 6) by (mod 2). Then $\tilde{\mathscr{U}}$ is a $\tilde{\Lambda}$-$\mathbb{Z}/8$-bimodule $\tilde{\mathscr{U}} = {}_{\tilde{\Lambda}}\tilde{\mathscr{U}}_{\mathbb{Z}/8}$.

Now denote by z_i (resp. $z_{\binom{v}{j}}$) the image of $z \in \mathbb{Z}$ in $\tilde{\mathscr{U}}_i$ (resp. in $\tilde{\mathscr{U}}_{\binom{v}{j}}$). For elements u, $v \in \tilde{\mathscr{U}}$ we write $u < v$ if there exists $a \in \tilde{\Lambda}$ with $au = v$. Then, in fact, all 2-primary elements from $\tilde{\mathscr{U}}_i$ (resp. $\tilde{\mathscr{U}}_{\binom{v}{j}}$) are linearly ordered as follows:

$$\begin{aligned} 1_1 &< 1_{\binom{3}{3}} < 1_{\binom{3}{1}} < 1_{\binom{3}{2}} < 1_{\binom{6}{3}} < 1_{\binom{6}{1}} < 1_{\binom{6}{2}} < 1_3 < 1_2 \\ &< 2_1 < 2_{\binom{3}{3}} < 2_{\binom{3}{1}} < 2_{\binom{3}{2}} < 2_{\binom{6}{3}} < 2_{\binom{6}{1}} < 2_{\binom{6}{2}} < 2_3 < 2_2 < 4_1. \end{aligned}$$

On the other hand, $1_5 < 1_4 < 4_1$ and $1_5 < 2_2$ and there are no other relations $<$ between 2-primary elements. Therefore we can proceed as in [4] and [5] to obtain the following list of indecomposable $\tilde{\mathscr{U}}$-matrices:

(a_1) with $a = 1,2,4$.

(a_i) with $i = 2,3$ and $a = 1,2$.

(1_i) with $i = 4,5$.

$(a_{\binom{v}{j}})$ with $a = 1,2$ and $\binom{v}{j}$ satisfying $v \in \{3,4,6\}$, $j = 1,2,3$ except $\binom{4}{3}$.

$\begin{pmatrix} a_i \\ 1_k \end{pmatrix}$ with $a = 1,2$ and $i = 1,2,3$ and $k = 4,5$ except $\begin{pmatrix} 2_2 \\ 1_5 \end{pmatrix}$.

$\begin{pmatrix} a_{\binom{v}{j}} \\ 1_k \end{pmatrix}$ with $a = 1,2$ and $k = 4,5$ and $\binom{v}{j}$ satisfying $v \in \{3,6\}$ and $j \in \{1,2,3\}$.

If we consider the 3-part of $\bar{\mathscr{U}}$-matrices the answer is easy. There is only one non-trivial indecomposable matrix (1_1) (isomorphic to (1_i) and $(1_{\binom{v}{j}})$ for all possible values of i, v, j). The Chinese remainder theorem implies that we can glue any 2-primary element from $\widetilde{\mathscr{U}}(P,H)$ with any 3-primary element with the same values of P and H. This gives us the following indecomposable $\mathscr{U}$-matrices (taking into account those "crossed out" under Morita-reduction).

(w_1) with $w = 1, \dots, 12$.

(w_i) with $i = 2,3$ and $w = 1, \dots, 6$.

$(w_{\binom{v}{j}})$ with $v = 1, \dots, 6$ and $j = 1,2,3$ and $w = 1, \dots, 6$. (1_4) and (1_5).

$\begin{pmatrix} w_i \\ 1_4 \end{pmatrix}$ with $i = 1,3$ and $w = 1, \dots, 6$.

$\begin{pmatrix} w_{\binom{v}{j}} \\ 1_4 \end{pmatrix}$ with $v = 1, \dots, 6$ and $j = 1,2,3$ and $w = 1, \dots, 6$.

$\begin{pmatrix} w_2 \\ 1_5 \end{pmatrix}$ with $w = 1,2,3$.

$\begin{pmatrix} w_i \\ 1_5 \end{pmatrix}$ with $i = 1,3$ and $w = 1, \dots, 6$.

$\begin{pmatrix} w_{\binom{v}{j}} \\ 1_5 \end{pmatrix}$ with $v = 1, \dots, 6$ and $j = 1,2,3$ and $w = 1, \dots, 6$.

$\begin{pmatrix} w_2 \\ 1_4 \end{pmatrix}$ with $w = 1,2,3$.

These indecomposables are in 1–1 correspondence with all elements in $\mathscr{L}_5 - \mathscr{L}_4 - \{S^5\}$. The correspondence is given as follows; compare the list of graphs following (1.1). The correspondence is easily deduced from the notation in (3.3) since the matrices describe the attaching map of the top cell.

$$(1_5) = (\eta)_3, \quad (1_4) = (\eta^2)_2, \quad (w_1) = (w)_1,$$

$$\begin{pmatrix} w_2 \\ 1_5 \end{pmatrix} = (\eta w \eta)_1^1, \quad \begin{pmatrix} w_3 \\ 1_4 \end{pmatrix} = (\eta^2 w \eta^2)_1^1, \quad \begin{pmatrix} w_3 \\ 1_5 \end{pmatrix} = (\eta^2 w \eta)_1^1,$$

$$\begin{pmatrix} w_2 \\ 1_4 \end{pmatrix} = (\eta w \eta^2)_1^1, \quad (w_2) = (\eta^2 w)_1^1, \quad \begin{pmatrix} w_1 \\ 1_4 \end{pmatrix} = (w\eta^2)_1^0,$$

$$(w_3) = (\eta w)_1^1, \quad \begin{pmatrix} w_1 \\ 1_5 \end{pmatrix} = (w\eta)_1^0, \quad (w_{\binom{v}{3}}) = (v\eta^2 w)_1^0,$$

$$\begin{pmatrix} w_{\binom{v}{2}} \\ 1_5 \end{pmatrix} = (\eta v \eta^2 w \eta)_0^1, \quad \begin{pmatrix} w_{\binom{v}{1}} \\ 1_4 \end{pmatrix} = (\eta^2 v \eta^2 w \eta^2)_0^1,$$

$$\begin{pmatrix} w_{\binom{v}{3}} \\ 1_5 \end{pmatrix} = (v\eta^2 w \eta)_0^0, \quad (w_{\binom{v}{2}}) = (\eta v \eta^2 w)_0^1,$$

$$\begin{pmatrix} w_{\binom{v}{3}} \\ 1_4 \end{pmatrix} = (v\eta^2 w \eta^2)_0^0, \quad (w_{\binom{v}{1}}) = (\eta^2 v \eta^2 w)_0^1,$$

$$\begin{pmatrix} w_{\binom{v}{1}} \\ 1_5 \end{pmatrix} = (\eta^2 v \eta^2 w \eta)_0^1, \quad \begin{pmatrix} w_{\binom{v}{2}} \\ 1_4 \end{pmatrix} = (\eta v \eta^2 w \eta^2)_0^1.$$

This completes the proof of the decomposition theorem (2.5). Q.E.D.

7. On the representation type of $\mathbf{F}^6$

Since we classified above the indecomposable homotopy types of $\mathbf{F}^5$ we can proceed to classify the homotopy types in $\mathbf{F}^6$. The method is similar to the computation in Section 3 for $\mathbf{F}^5$. Similarly as in (3.1) (6) we now obtain for $X \in \mathbf{F}^6$ a homomorphism

$$f: H_6 X \to \Gamma_5(X^5), \tag{7.1}$$

where we may assume that the skeleton X^5 again is given by a homology decomposition of X. Moreover, X^5 is a one-point union of indecomposable objects in $X(\mathscr{L}_5)$ by (2.5). Hence we can compute $\Gamma_5(Y)$ for each object Y in $X(\mathscr{L}_5)$ in order to obtain an explicit form of $\Gamma_5(X^5)$. Then we have to understand the action of the group of homotopy equivalences of X^5 on the $\Gamma_5 X^5$ and using this action we have to construct a "normal form" of (7.1).

7.2. Theorem. *For $Y \in X(\mathscr{L}_5)$ the group $\Gamma_5 Y$ is either given by* (3.4) *or by the following list*:

$\Gamma_5 X(g) = 0$	*for* $g = S^0, (v\eta^2)_0^0, (v\eta^2 w)_0^0, (v\eta^2 w\eta)_0^0$,
$\Gamma_5 X(g) = \mathbb{Z}/24$	*for* $g = (\eta)_0, (\eta v\eta^2)_0^1, (\eta v\eta^2 w\eta)_0^1, (\eta v\eta^2 w)_0^1$,
$\Gamma_5 X(g) = \mathbb{Z}/12$	*for* $g = (v\eta)_0^0, (v\eta^2 w\eta^2)_0^0$,
$\Gamma_5 X(g) = \mathbb{Z}/2$	*for* $g = (\eta^2)_0, (v)_0, (\eta^2 v\eta^2)_0^1, (\eta^2 v\eta^2 w)_0^1, (\eta^2 v\eta^2 w\eta)_0^1$,
$\Gamma_5 X(g) = \mathbb{Z}/24 \oplus \mathbb{Z}/12$	*for* $g = (\eta v\eta)_0^1, (\eta v\eta^2 w\eta^2)_0^1$,
$\Gamma_5 X(g) = \mathbb{Z}/2 \oplus \mathbb{Z}/12$	*for* $g = (\eta^2 v\eta)_0^1, (\eta^2 v\eta^2 w\eta^2)_0^1$,
$\Gamma_5 X(g) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$	*for* $g = (\eta^2 v)_0^1$,
$\Gamma_5 X(g) = \mathbb{Z}/24 \oplus \mathbb{Z}/2$	*for* $g = (\eta v)_0^1$.

The theorem shows that there are exactly 328 elements $g \in \mathscr{L}_5$ for which $\Gamma_5(g)$ is nonzero. This shows that the 23×23 matrix in (5.1) has to be replaced by a 328×328 matrix with the additional complication that various $\Gamma_5(Y)$ in (7.2) are given by the direct sum of two cyclic groups. We now consider the special case that X^5 in (7.1) is a one-point union of d copies of $X(\eta^2 v)_0^1$ with $v_0 \in \{1, \dots, 6\}$. In this case we get the following problem.

Let $H = \mathbb{Z}^h$ and $V = \mathbb{Z}^d$ be finitely generated free abelian groups and consider homomorphisms

$$f, f' : H \to V \otimes \mathbb{Z}/2 \oplus V \otimes \mathbb{Z}/2. \tag{7.3}$$

Then f is equivalent to f' if there exist automorphisms $n_0, n_3, n_4 \in \mathrm{Aut}(V)$ and $h \in \mathrm{Aut}(H)$ with

$$\begin{aligned} v_0 n_0 &\equiv v_0 n_0 \pmod{12}, \\ n_0 &\equiv n_3 \pmod 2 \end{aligned} \tag{56}$$

such that $f' = (n_3 \oplus n_4) \otimes \mathbb{Z}/2 \circ f \circ h$. Equivalence classes of such homomorphisms are in 1–1 correspondence with homotopy types of CW-complexes of the form

$$\underbrace{X(\eta^2 v)_0^1 \vee \cdots \vee X(\eta^2 v)_0^1}_{d\text{-times}} \cup_f \underbrace{e^6 \cup \cdots \cup e^6}_{h\text{-times}}. \tag{57}$$

7.4. Proposition. *There are infinitely many indecomposable homotopy types in* $\mathbf{F}^6$ *and* $K_0(\mathbf{F}^6) = \mathbb{Z}^\infty$.

Proof. We consider the decomposition of homotopy types as in (7.3) (57) for $v_0 = 2$. Then (7.3) (56) shows that the 2-primary part of this problem has the representation type of the well-known Kronecker quiver

$$\bullet \rightrightarrows \bullet,$$

which has tame representation type and infinitely many indecomposable representations over $\mathbb{Z}/2$. Q.E.D.

Proposition (7.4) together with theorem (2.5) yields a proof of Theorem B in the introduction.

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